# TECHNICAL NOTE
## D-604

## THE EFFECT OF RADIATION FORCE ON SATELLITES OF CONVEX SHAPE

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(The author is grateful to Dr. Willi Heybey for continual cooperation and many helpful discussions of the mechanical aspects of the subject. He also wishes to thank Mr. James E. Mabry who carried out part of the numerical computations.)

## SUMMARY

Computations of satellite position and attitude will have to take into account the pressure exerted by the solar radiation which, in general, results both in a force and a torque acting on the body.

The well-known value of the solar constant serves for the computation of the integrated radiation force which is given the form

$$F = p_0 \cdot A \cdot C_F$$

in analogy to aerodynamic forces (A = projected area; $p_0$ = radiation pressure in the vicinity of the earth). The "radiation force coefficient" $C_F$ has been determined, in the present study, for several convex bodies and was found within the limits

$$0 < C_F \leqq 2$$

Since part of the radiation force is caused by the reflection of light the optical properties of common materials, as transparency, absorption, reflectivity, have been reviewed. Their dependence on angle of incidence and wave-length complicates the computation of $C_F$. However, adequate values cannot be obtained from geometry alone without due regard to the physics involved in the reflection process.

# I. INTRODUCTION

## A. General

Some of the exterior forces that will or may act on a satellite are provided by:

Gravitation

Air Drag

Radiation Pressure

Meteoric Impacts

Cosmic Rays

The Magnetic Field of the Earth

Orbit calculations are largely based on the gravitational and aerodynamic forces. Attitude control and determination of position, however, require the study of the remaining forces.

The object of the present report is to investigate the influence of the radiation pressure of the sun's light.

When studying the radiation effect, we have to consider separately the influence of the incoming light and that of the reflected light. The latter depends, in a large measure, on the reflectivity of the opaque surface material. For transparent media, the transparency also has to be considered. All bodies, however, show a certain degree of reflectivity. Both the reflectivity and the transparency are functions of the angle of incidence of light, of the wave-length, and of the refractive index. These facts complicate our present problem. It will be necessary to derive exact formulas for the force produced by the radiation. In analogy to the aerodynamic equation for the drag, the radiation force, F, acting on a body will be expressed in the form:

$$F = p_0 \cdot A \cdot C_F , \qquad (1)$$

where A means the projected area of the body in the direction of the light, and $p_o$ is the radiation pressure of the sun's light in the vicinity of the earth. We can consider both $p_o$ and A as given and, therefore, it is necessary to derive formulas for the coefficient, $C_F$, which we refer to as the radiation force coefficient. Determination of

this coefficient $C_F$ is the first step in the studies concerning the exterior forces acting on satellites. The present report will especially deal with the case of convex-shaped bodies. A second publication on concave-shaped bodies is in preparation.

For all computations, some physical data have to be known, especially the reflectivity. A few useful tables are given in this report. For some general purpose a set of tables containing the values of the reflectivity in the "Gaussian plane of the indices of refraction" is planned.

B. The Radiation of the Sun

In this report, only the wave radiation of the sun in the vicinity of the earth is considered. Corpuscular radiation will be excluded. Since the energy curve of the sun's wave radiation includes an interval stimulating the sensibility of the human eye, the sun's radiation is mainly felt as light and, therefore, the expression "light pressure" is often used instead of "radiation pressure."

The total radiation of the sun amounts to approximately 3.78 X $10^{33}$ erg/sec. The radiation emitted by the sun extends from short ultra-violet to long infra-red wave-lengths. According to its absorption spectrum, the sun is a GO-star with the maximum radiation at the wave-length 4780 angstroms. Diagrams that show the distribution of energy over the whole spectrum are well known. (Ref. 1)

Only a portion of the sun's output reaches the vicinity of the earth. The free space intensity of the solar radiation flowing per unit time through a surface of 1 $cm^2$ placed normally to the light direction of the earth's mean solar distance (1,496 x $10^{13}$ cm) is known as the solar constant of radiation, S, and has the value (Ref. 2)

$$\begin{aligned} S &= 1.97 \pm 0.01 \text{ cal cm}^{-2} \text{ min}^{-1} \\ &= 1.374 \cdot 10^6 \text{ erg cm}^{-2} \text{ sec}^{-1} \end{aligned} \tag{2}$$

If the radiation is to be considered over a long time period, it is necessary to include the change in distance between the earth and the sun. Because of the eccentricity, e, of the orbit of the earth the intensity, J, of the radiation varies by 7% between perihel and aphel:

$$\frac{J_p}{J_a} = \left(\frac{1 + e}{1 - e}\right)^2 = \left(\frac{1.017}{0.983}\right)^2 = 1.07 \tag{3}$$

The values of the energy of the sun's radiation at the extreme distances are:

| | Earth-Sun Distance in cm | Energy of Sun's Radiation in erg $cm^{-2}$ $sec^{-1}$ |
|---|---|---|
| Perihel | $1.4710 \times 10^{13}$ | $1.351 \times 10^{6}$ |
| Aphel | $1.5210 \times 10^{13}$ | $1.397 \times 10^{6}$ |

The earth-satellite distance is small when compared to the earth-sun distance. For example, in the case of a satellite at a distance of nearly 6 earth radii, the error in radiation intensity is only 0.12% using the earth-sun distance instead of the sun-satellite distance.

In order to compute the effect of the radiation on a satellite, the radiation of subdivided parts of the spectrum is needed. In Table I the distribution of energy in the spectrum of the sun is given, taken from Linke's "Meteorologisches Taschenbuch." (Ref. 3) This table gives the sun's energy in cal $cm^{-2}$ $min^{-1}$ per 100 angstroms. The values of Table I are plotted in Figure 1, curve "a." The scale is

| | | |
|---|---|---|
| Abscissa | 1.00 cm | $0.2 \times 10^{-4}$ cm |
| Ordinate | 1.00 cm | $2 \cdot 10^3$ cal $cm^{-3}$ $min^{-1}$ |

and hence 1 $cm^2$ of the graph is equal to 0.04 cal $cm^{-2}$ $min^{-1}$. Thus the total area under curve "a" represents the value S = 1.98 cal $cm^{-2}$ $min^{-1}$.

From Table I another graph was established, Figure 2, in which curve "a" gives the amount of the solar constant from $\lambda = 0$ until a desired wave-length is reached or for any wave-length interval by using the difference in the values of two ordinates.

The sun's radiation stems from a surface region whose temperature is nearly 6000° K. As the sun's radiation sometimes is likened to that of a black body, it is of interest to plot the distribution of radiation density for a black body of the temperature 6000° K, which is given by Planck's law (Appendix).

$$J d\lambda = c_1 \lambda^{-5} \left(e^{c_2/\lambda T} - 1\right)^{-1} d\lambda \left[\frac{\text{Watt}}{cm^2}\right] \tag{4}$$

In Figrue 1, curve "b" shows Planck's radiation function for a black body of 6000° K. Curve "b" in Figure 2 presents the portions of the total radiation of a black body (6000° K) in the same way as does curve "a" for the solar radiation.

C. The Mechanical Force Exerted by Radiation

In the vicinity of the earth,the sun's light is to be considered as a system of plane, finite, electromagnetic waves, traveling in empty space. By Maxwell's electromagnetic theory, an impulse in the direction of the beam is associated to each wave. Let the quantity J [Watt $cm^{-2}$] denote the intensity of the light waves per time and surface units. When the wave of light meets the surface of a body perpendicularly, the radiation pressure is given by

$$p = \frac{J}{c} \tag{5}$$

The value of J is equal to the magnitude of the Poynting vector, $\bar{E} \times \bar{H}$, where $\bar{E}$ and $\bar{H}$ are the electromagnetic field vectors. We measure the electromagnetic quantities in the so-called Giorgi system, as employed, e. g., in reference (4). Therefore $p_o$, the radiation pressure at the earth's mean solar distance, is given by

$$\begin{aligned} p_o &= \frac{J_o}{c} = \frac{|\bar{E} \times \bar{H}|}{c} = \frac{S}{c} \\ &= \frac{0.14 \cdot 10^7}{3 \cdot 10^{10}} \cdot \frac{erg}{cm^3 \cdot sec} \\ &= 4.7 \cdot 10^{-5} \frac{erg}{cm^3 \cdot sec} \\ &= 4.7 \cdot 10^{-5} \frac{dyne}{cm^2} \end{aligned} \tag{6}$$

The force due to radiation of normal incidence acting on a plane surface element $dA_o$ is:

$$dF = p_o \cdot dA_o \tag{7}$$

If the light rays form an angle, $\alpha$, with the surface normal, the force is given by:

$$dF = p_o \cdot dA_o \cdot \cos\alpha \tag{8}$$

It will produce a translatory motion, which moves a body away from the sun. If the material of the body possesses a transparency, only that part of the light which is not allowed to pass through will contribute to the radiation force. The transparency is a function of the angle of incidence and of the refractive index of the material which in turn depends on the wave-length. It should therefore be written as:

$$T = T(n(\lambda), \alpha)$$

With a partly transparent surface material the force element is smaller than before:

$$dF = p_0 \cdot dA_0 \cdot (1 - T(n(\lambda), \alpha)) \cdot \cos\alpha. \qquad (9)$$

In satellites we encounter shapes like spheres, cylinders, cones, and planes. Most satellites are composed of parts of such bodies, e. g., the communication satellites. Antennas in use are of spherical and parabolic shape. Only in special areas the radiation will act normal to the surface. If the surface is not normal to the radiation, there are force components due to the reflectivity that give lateral displacements.

The reflectivity of a material is a function of the index of refraction, i. e., of the wave-length, and of the angle of incidence; thus we write:

$$R = R(n(\lambda), \alpha).$$

In the following discussion "ideal reflection" will be understood to mean that the beam follows the ordinary reflection law, and that it experiences no loss in energy (R = 1.)

In summary, it can be stated that the radiation force acting on a satellite is a function of the following quantities:

1. The sun-satellite distance, $s$,
2. The size and the shape of the surface,
3. The optical properties: reflective ratio and transparency ratio,
4. The angle of incidence of light, $\alpha$,
5. The distribution of the radiation energy in the solar spectrum.

Before we study the effect of radiation pressure on selected bodies of different shapes we will give a survey about some facts in optics, which will be needed for a proper evaluation.

## D. Optical Properties

### a. The Reflecting Power of Transparent or Dielectrical Media

The oscillation of the incident light may be assumed to consist of two equal components that are plane-polarized in mutually perpendicular azimuths: one perpendicular to the plane of incidence and the other in the plane of incidence.

The following symbols are used:

| | | |
|---|---|---|
| $E_{\perp}$ , $E_{\parallel}$ | : | Amplitudes of the electric field intensity perpendicular and parallel to the plane of incidence |
| J | : | Intensity of the light |
| i, r, t | : | Indices referring to incident, reflected and transmitted beam |
| $\beta$ | : | Angle of refraction |

The amplitudes of the electric field intensity are given by Fresnel's reflection equations:

$$
\begin{aligned}
E_{\perp}^{r} &= -E_{\perp}^{i} \cdot \frac{\sin(\alpha-\beta)}{\sin(\alpha+\beta)} \\
E_{\parallel}^{r} &= E_{\parallel}^{i} \cdot \frac{\tan(\alpha-\beta)}{\tan(\alpha+\beta)} \\
E_{\perp}^{t} &= E_{\perp}^{i} \cdot \frac{2\sin\beta\cdot\cos\alpha}{\sin(\alpha+\beta)} \\
E_{\parallel}^{t} &= E_{\parallel}^{i} \cdot \frac{2\sin\beta\cdot\cos\alpha}{\sin(\alpha+\beta)\cos(\alpha-\beta)}
\end{aligned}
\qquad (10)
$$

From these intensities become:

$$J_{\perp}^{r} = J_{\perp}^{i} \cdot \frac{\sin^{2}(\alpha-\beta)}{\sin^{2}(\alpha+\beta)}$$

$$J_{\parallel}^{r} = J_{\parallel}^{i} \cdot \frac{\tan^{2}(\alpha-\beta)}{\tan^{2}(\alpha+\beta)}$$

$$J_{\perp}^{t} = J_{\perp}^{i} \cdot \frac{\sin 2\alpha \cdot \sin 2\beta}{\sin^{2}(\alpha+\beta)} \tag{11}$$

$$J_{\parallel}^{t} = J_{\parallel}^{i} \cdot \frac{\sin 2\alpha \cdot \sin 2\beta}{\sin^{2}(\alpha+\beta) \cdot \cos^{2}(\alpha-\beta)}$$

The reflectivity, R, and the transparency, T, are defined by:

$$R = \frac{\dfrac{J_{\perp}^{r}}{J_{\perp}^{i}} + \dfrac{J_{\parallel}^{r}}{J_{\parallel}^{i}}}{2} \tag{12}$$

$$T = \frac{\dfrac{J_{\perp}^{t}}{J_{\perp}^{i}} + \dfrac{J_{\parallel}^{t}}{J_{\parallel}^{i}}}{2} \tag{13}$$

The angle $\beta$ is related to $\alpha$ by Snell's law:

$$\frac{\sin\alpha}{\sin\beta} = n, \tag{14}$$

where n is the refractive index.

There is a special case for $\tan\alpha = n$ (Brewster's law), where $E^r_{\parallel}$ becomes zero and all of the reflected light oscillates perpendicularly to the plane of incidence. For normal incidence (angle $\alpha = 0$, therefore, $\beta = 0$), the equations of reflection and transparency can be transformed into the expressions:

$$R = \left(\frac{n-1}{n+1}\right)^2$$

$$T = \frac{4n}{(n+1)^2} \qquad (15)$$

R becomes unity only when n is infinite, which is the case of an ideal reflector. Materials with infinite refractive indices have an electrical conductivity and will be considered in the next section.

From the equations (12) the reflectivity can be calculated. Figure 3 shows the variation of reflecting power with the angle of incidence for crown glass (n = 1.5): curve $a_{\parallel}$ for light polarized in the plane of incidence, curve $a_{\perp}$ for light polarized perpendicularly to the plane of incidence. Since it was assumed at the very beginning of this section that, for incident light, the two intensity components are equal, it is assumed in the figure that

$$J_{\perp}^{i} = J_{\parallel}^{i} = 1 \qquad (16)$$

b. <u>The Reflecting Power of Absorbing Materials or Electrical Conductors</u>

These materials include metals and alloys. Instead of the real refractive index n it is convenient here to use a complex refractive index $N = n - ki = n(1 - \varkappa i)$. In this expression, n is no longer defined in the same way as in Snell's law. This law is still effective when n is replaced by N, but it has only a formal meaning. The angle $\beta$ now assumes a complex value, $\beta = \varphi + \chi i$. Fresnel's equations for the intensity of light change to (Ref. 5):

$$\frac{J_{\perp}^{r}}{J_{\perp}^{i}} = \frac{\sin^2(\alpha-\varphi) + \sinh^2\psi}{\sin^2(\alpha+\varphi) + \sinh^2\psi}$$

$$\frac{J_{\parallel}^{r}}{J_{\parallel}^{i}} = \frac{\left[\sin^2(\alpha-\varphi)+\sinh^2\psi\right]\cdot\left[\cos^2(\alpha+\varphi)+\sinh^2\psi\right]}{\left[\sin^2(\alpha+\varphi)+\sinh^2\psi\right]\cdot\left[\cos^2(\alpha-\varphi)+\sinh^2\psi\right]} \qquad (17)$$

where the angles $\varphi$ and $\psi$ are evaluated from:

$$\sinh^2\psi = -\frac{c}{2} + \sqrt{b^2 + \frac{c^2}{4}}$$

$$\cos\varphi = \frac{b}{\sinh\psi} \qquad (18)$$

$$b = \frac{\kappa \sin\alpha}{n(1+\kappa^2)} = \frac{k \sin\alpha}{n^2+k^2}$$

$$c = 1 - \frac{\sin^2\alpha}{n^2(1+\kappa^2)} = 1 - \frac{\sin^2\alpha}{n^2+k^2}.$$

The quantities $\kappa$ and $k$ in the expression for N are termed as follows:

$\kappa$ = extinction or absorption index

$k$ = extinction or absorption coefficient

These quantities can be determined, e.g., by measuring the light intensity, J, transmitted through thin films of the opaque material. The following formulas are found to hold:

$$J = J_0 \cdot e^{-4\pi \kappa d/\lambda_1}$$
$$J = J_0 \cdot e^{-4\pi k d/\lambda} \qquad (19)$$

with

$$\frac{\lambda}{\lambda_1} = n,$$

where

$\lambda$ wave-length in vacuo

$\lambda_1$ wave-length inside the medium

$d$ the thickness of the medium.

As in the case of dielectrical substances the reflectivity is given by a particularly simple expression, if $\alpha = \beta = 0$ (normal incidence):

$$R = \frac{J_{\parallel}^{k}}{J_{\parallel}^{i}} = \frac{J_{\perp}^{k}}{J_{\perp}^{i}} = \frac{(n-1)^2 + n^2\kappa^2}{(n+1)^2 + n^2\kappa^2} \qquad (20)$$

While the index of refraction of a transparent medium varies only slightly with the wave-length of light, the index for metals shows marked dependence on wave-length. A great number of determinations have been made on all kinds of materials and the data are published in many physical handbooks, e. g., (Ref. 6). Values for the reflectivity of light are also tabulated. But the great majority of the tables contain only the reflection for normal incidence of light. Values for oblique incidence seem to be very rarely published. Generally, it will be noticed from available information that the reflectivity of most conducting materials is low in the ultra-violet and increases with the wave-length to approximately 90% or more. In order to give a survey on the reflectivity and the index of refraction, two graphs were made. Figure 4 shows, in the Gaussian plane, the index of refraction, $N = n - ki$, for a number of metallic elements; the values are taken from Reference 7. The abscissa denotes the real part of the refractive index, and the ordinate the imaginary part. The different curves refer to selected metals and show how their index N changes with the wave-length. The three-place values at the beginning and end of the curves give the wave-length in $m\mu$ ($= 10^{-5}$ cm). In order not to overcrowd the graph, only some samples of elements are shown. From equation 20 the reflectivity at zero angle of incidence was computed and plotted in Figure 5. The coordinates are the same as in Figure 4 and the curves give the reflectivity in 0.1 intervals. If the reflection power with incident angle other than $0^{o}$ is required, a distinction must be made between the light oscillating in

the plane of incidence and perpendicularly to it. Therefore, for each angle of incidence, two diagrams like Figure 5 are necessary, where again the same coordinate system as in Figure 4 will be used. The refractive index N depends both on the material and wave-length of the incident light. For each material and wave-length two curves, such as given for stainless steel ($\lambda$ = 436 m$\mu$, N = 0.9 - 2.5 i) and crown glass in Figure 4 have to be drawn. From these curves the reflectivity for each angle of incidence is obtainable. The computation of the reflectivity for all materials would require much labor and would be hardly economical. However, it is planned to work out diagrams such as Figure 5 for $\alpha$ = 0, 10, 20,........80, 85, 90°, since this can be done without reference to specific materials. The results will be given in a special report. From these graphs one can extract the values needed to obtain the curve of the reflecting power for all possible refractive indices. Such tables apparently are not published as yet, and it is felt that they would be of great help in engineering work.

Some remarks with respect to the refraction indices are necessary. Physical tables give the values of optical constants with great accuracy, and the values of the refractive indices of transparent media are presented to seven decimal places. But the determination of optical constants such as the index of refraction requires experiments which are among the most difficult ones in physics. In addition materials are not easily prepared or become dull within a short time.

Thus, the problem may become almost unsolvable in some instances. In conclusion we may say that values of the refractive index with an accuracy of one per cent are desirable. Nevertheless, the values published for opaque materials are useful to get an idea of the order of magnitude of the reflectivity for different light frequencies. In the case of newly developed materials used for satellites, special measurements of the reflectivity will be necessary.

## II. THE EFFECT OF RADIATION ON BODIES OF DIFFERENT SHAPES

Among the surfaces, the plane surface element evidently is the most important. In the following sections we will consider the element and the plane itself. We shall also deal with bodies which are symmetric with respect to the light direction. Such bodies can have convex or concave form. In this report only bodies with a convex surface will be considered. A future report will deal with the behaviour of the radiation pressure on concave bodies, among which the parabolic and the spherical mirrors are the most important. Finally, cylindrically shaped bodies are of interest, and we shall study here the right circular cylinder whose axis of symmetry is perpendicular to the direction of light.

2X



A. The Surface Element

We consider a surface element $dA_0$ with the exterior normal $\bar{n}$ located at the origin of a rectangular coordinate system, x, y, z, with unit vectors $\bar{i}$, $\bar{j}$, $\bar{k}$. The light travels in the negative y-direction; after the reflection on the surface element, it has the direction $\bar{s}$. From geometric optics it follows that the incident angle $\alpha$, the angle between $\bar{n}$ and $-\bar{j}$, is the same as the angle between $\bar{n}$ and $\bar{s}$, and that the three vectors are in one plane, called the plane of incidence. This plane cuts the plane y = o in a line which makes an angle, $\gamma$, with the z-axis, Figure 6.

The acting radiation force we shall write in the form $\bar{F} = x\bar{i} + y\bar{j} + z\bar{k}$. Then the force element, produced by the incoming light emerges as:

$$d\bar{F}^i = -p_0 \cdot (\bar{j}\bar{n})\, dA_0 \cdot (1 - T(\alpha))\, \bar{j}$$
$$= -p_0 \cdot \cos\alpha\, dA_0 \cdot (1 - T(\alpha))\, \bar{j} \qquad (21)$$

The radiation force produced by the reflected light works in the opposite direction of the reflected beam and becomes:

$$d\bar{F}^r = -p_0 (\bar{s}\bar{n})\, dA_0 \cdot R(\alpha) \cdot \bar{s} \qquad (22)$$

Since

$$\bar{s} = \sin\gamma \sin 2\alpha\, \bar{i} + \cos 2\alpha\, \bar{j} + \cos\gamma \sin 2\alpha\, \bar{k} \qquad (23)$$

and

$$\bar{n} = \sin\gamma \sin\alpha\, \bar{i} + \cos\alpha\, \bar{j} + \cos\gamma \sin\alpha\, \bar{k}, \qquad (24)$$

$d\bar{F}^r$ is given by:

$$d\bar{F}^{R} = -p_0 \cos\alpha \cdot dA_0 \cdot \mathcal{R}(\alpha) \cdot \left(\sin\gamma \sin 2\alpha\,\bar{i} + \cos 2\alpha\,\bar{j} + \cos\gamma \sin 2\alpha\,\bar{k}\right). \tag{25}$$

In the differentials for the radiation force, given by the equations (21 and 25), the functions T ($\alpha$) and R ($\alpha$) are still unknown. Values of T ($n(\lambda), \alpha$) and R ($n(\lambda), \alpha$) for given N, n, and $\alpha$ can be calculated by the equations (10) through (18). The characteristics of both functions are the same and in order to discuss them we use only R. We consider a surface element for which the angle interval $\alpha_{a-1} \leq \alpha \leq \alpha_a$ is small enough so that the reflectivity is only a function of the wave-length of the light. The part of the reflected intensity which belongs to the wave-length $\lambda$ is written as R ($n(\lambda), \alpha$) $\cdot J_\lambda\, d\lambda$, where $J_\lambda$ refers to the intensity of the incident light. Then the quantity R ($\alpha$) obtained by the integration of the reflectivity over the whole energy spectrum:

$$\mathcal{R}(\alpha) = \frac{\int_0^\infty R\left[n(\lambda), \alpha\right] \cdot J_\lambda\, d\lambda}{\int_0^\infty J_\lambda\, d\lambda} \tag{26}$$

gives the reflectivity associated with the angle $\alpha$ in the above range. In order to master this integration the radiation is divided into d$\lambda$-intervals in which values of R are represented by $\overline{R_\rho}(\alpha)$ and given by:

$$\overline{R_\rho(\alpha)} = \frac{1}{\lambda_\rho - \lambda_{\rho-1}} \int_{\lambda_{\rho-1}}^{\lambda_\rho} R_\rho(\alpha)\, d\lambda$$

Thus R ($\alpha$) becomes:

$$R(\alpha) = \frac{\sum_{\rho=1}^{\mu} \overline{R_\rho(\alpha)} \int_{\lambda_{\rho-1}}^{\lambda_\rho} J_\lambda\, d\lambda}{\int_0^\infty J_\lambda\, d\lambda} \tag{27}$$

with $\lambda_0 = 0$ and $\lambda_\mu = \infty$.

In the same way we introduce a transparency function T ($\alpha$):

$$T(\alpha) = \frac{\sum_{\tau=1}^{\mu} \overline{T_\tau(\alpha)} \int_{\lambda_{\tau-1}}^{\lambda_\tau} J_\lambda \, d\lambda}{\int_0^\infty J_\lambda \, d\lambda}, \tag{28}$$

where

$$\overline{T_\tau(\alpha)} = \frac{1}{\lambda_\tau - \lambda_{\tau-1}} \int_{\lambda_{\tau-1}}^{\lambda_\tau} T_\tau(\alpha) \, d\alpha . \tag{29}$$

In order to calculate these functions we can use tables or graphs, such as given by Figures 2 and 4.

B. The Body of Arbitrary Shape

The force acting on an arbitrary body is obtained by integration of the forces which act on the surface elements. Thus we have:

$$|-\bar{F}^i| \equiv F^i = \int_{\text{Surface } A_0} dF^i = p_0 \int_{\text{Surface } A_0} [1 - T(\alpha)] \cos\alpha \, dA_0 \tag{30}$$

The surface element $dA_0$ will depend on two variables, one of which we can often take as the angle of incidence, $\alpha$, whereas the other one will be determined by the shape of the body we consider. Let this variable be denoted by $\xi$. In most cases, the surface element can then be written as:

$$dA_0 = g(\alpha, \xi) \, d\alpha \, d\xi , \tag{31}$$

so that

$$F^i = p_0 \int_\alpha \int_\xi [1 - T(\alpha)] \, g(\alpha, \xi) \cos\alpha \, d\xi \, d\alpha \tag{32}$$

For short we put:

$$\int_{\xi} g(\alpha, \xi)\, d\xi = G(\alpha). \tag{33}$$

Thus the force takes on the form:

$$F^{i} = p_o \int_{\alpha} [1-T(\alpha)] \cdot G(\alpha) \cdot \cos\alpha\, d\alpha \tag{34}$$

But the values of the functions T ($\alpha$) and R ($\alpha$) are given only for angle intervals $\alpha_{a-1} \leqq \alpha \leqq \alpha_a$ . Therefore, we have to sum the integrals over all angles $\alpha$ :

$$F^{i} = p_o \sum_{t=1}^{\nu} \int_{\alpha_{t-1}}^{\alpha_t} (1-T_t) \cdot G(\alpha) \cos\alpha\, d\alpha, \tag{35}$$

where

$$T_t = \frac{1}{\alpha_t - \alpha_{t-1}} \int_{\alpha_{t-1}}^{\alpha_t} T(\alpha)\, d\alpha. \tag{36a}$$

Analogously

$$R_r = \frac{1}{\alpha_r - \alpha_{r-1}} \int_{\alpha_{r-1}}^{\alpha_r} R(\alpha)\, d\alpha. \tag{36b}$$

The integral becomes dimensionless by dividing through the projected area $A\,[cm^2]$ as seen from the direction of the incoming light:

$$\bar{F}^{i} = -\bar{j} \cdot p_o \cdot A \cdot \frac{1}{A} \sum_{t=1}^{\nu} \int_{\alpha_{t-1}}^{\alpha_t} (1-T_t) \cdot G(\alpha) \cdot \cos\alpha\, d\alpha \tag{37}$$

In the same way, the radiation force of the reflected light becomes:

$$\bar{F}^{r} = -p_0 \cdot A \cdot \frac{1}{A} \cdot \sum_{k=1}^{\nu} \int_{\alpha_{k-1}}^{\alpha_k} R_k \cdot G(\alpha) \cdot \tag{38}$$

$$\cdot \left( \sin\gamma \sin 2\alpha \, \bar{i} + \cos 2\alpha \, \bar{j} + \cos\gamma \sin 2\alpha \, \bar{k} \right) d\alpha .$$

We introduce the notation $C_F$ and write the force components as:

$$F^{i} = p_0 \cdot A \cdot C_F^{i}$$

$$F^{r} = p_0 \cdot A \cdot C_F^{r} \tag{39}$$

The coefficients $c_F$ are given by:

$$C_F^{i} = \frac{1}{A} \sum_{t=1}^{\nu} \int_{\alpha_{t-1}}^{\alpha_t} (1 - T_t) \cdot G(\alpha) \cdot \cos\alpha \, d\alpha$$

$$C_F^{r} = \sqrt{\left(C_{F_x}^{r}\right)^2 + \left(C_{F_y}^{r}\right)^2 + \left(C_{F_z}^{r}\right)^2}$$

$$C_{F_x}^{r} = \frac{1}{A} \sum_{k=1}^{\nu} \int_{\alpha_{k-1}}^{\alpha_k} G(\alpha) \cdot R_k \cdot \cos\alpha \sin\gamma \sin 2\alpha \, d\alpha \tag{40}$$

$$C_{F_y}^{r} = \frac{1}{A} \sum_{k=1}^{\nu} \int_{\alpha_{k-1}}^{\alpha_k} G(\alpha) \cdot R_k \cdot \cos\alpha \cos 2\alpha \, d\alpha$$

$$C_{F_z}^{r} = \frac{1}{A} \sum_{k=1}^{\nu} \int_{\alpha_{k-1}}^{\alpha_k} G(\alpha) \cdot R_k \cdot \cos\alpha \cos\gamma \sin 2\alpha \, d\alpha .$$

The equations (39), $F = c_F \times A \times p_o$, (41)

have the shape of the aerodynamic equation for the drag:

$$D = C_D \times A \times q, \tag{42}$$

($C_D$ = drag coefficient, q = dynamic pressure).
Therefore, $c_F$ might be termed the "radiation force coefficient"; it is a function of the shape and of the optical properties of the body considered.

C. The Surface Element of a Symmetrical Body

In the following we consider only such bodies which are symmetrical relative to the light direction, in the sense that the individual force components normal to the incident light cancel each other, e. g., spherical and parabolic surfaces and the cone and cylinder in suitable orientations. The only force component is then in (-j) direction. One exception will be the plane surface inclined towards the light direction.

Let $dA_o$ in Figure 7 be a surface element of any symmetrical body; let the plane of incidence be the drawing plane. The element of the projected area, dA, becomes:

$$dA = dA_o \cdot \cos\alpha$$

Then the acting force, dF, is given by:

$$dF = dF_y^i + dF_y^r = p_o \cdot dA \cdot C_F$$

$$= p_o\, dA_o \left[1 - T(\alpha)\right] \cos\alpha + p_o\, dA_o \cdot R(\alpha) \cos 2\alpha \cos\alpha$$

The radiation force coefficients, $C_F^i$, and $C_F^r$, are:

$$C_F^i = 1 - T(\alpha) \tag{44}$$

$$C_F^r = R(\alpha) \cdot \cos 2\alpha .$$

Special values for the radiation force coefficient are determined as follows:

Opaque Materials

1. Medium with total absorption

$$\mathcal{R}(\alpha) = 0 \,, \quad T(\alpha) = 0$$

$$C_F = 1$$

2. Medium with ideal reflection

$$\mathcal{R}(\alpha) = 1 \qquad T(\alpha) = 0$$

$$C_F = 1 + \cos 2\alpha = 2 \cos^2 \alpha \,.$$

3. Medium with partial reflection

$$0 < \mathcal{R}(\alpha) < 1 \,, \quad T(\alpha) = 0$$

$$C_F = 1 + \mathcal{R}(\alpha) \cdot \cos 2\alpha \,.$$

Transparent Materials

4. Medium without absorption

$$\mathcal{R}(\alpha) + T(\alpha) = 1$$

$$\begin{aligned} C_F &= 1 - T(\alpha) + \mathcal{R}(\alpha) \cos 2\alpha \\ &= \mathcal{R}(\alpha) \cdot (1 + \cos 2\alpha) \\ &= 2 \mathcal{R}(\alpha) \cdot \cos^2 \alpha \,. \end{aligned}$$

From the above four cases it can be seen that the radiation force coefficient attains the value 2 only at normal incidence of light and for ideally reflecting materials (case 2). At an incident angle $\alpha = 45^{\circ}$ the reflected light yields no force component in the direction of the incident light. We then have $C_F = 1$ for all opaque materials. For angles of incidence $\alpha > 45^{\circ}$ all reflecting materials have a force component opposite to the light direction; $C_F$ is smaller than unity and

goes to zero with $\alpha \rightarrow \frac{\pi}{2}$.

D. The Plane Surface (Solar Sail)

We consider a plane disc which may serve as an idealized model of a solar sail, the flight dynamics of which will be discussed briefly. The position of this plane body is indicated in Figure 8.

In the absence of reflectivity, the disc will be accelerated in the direction of the sun's light independent of inclination. Therefore, only drag is present. A reflecting disc also has a lift component. From the equations (21) and (25) where $\gamma$ can be taken as zero the acting force on a solar sail becomes:

$$\bar{F} = \bar{F}_y + \bar{F}_z = p_0 \cdot A\left(-C_D^i \bar{j} - C_D^k \bar{j} + C_L^k \bar{k}\right) \quad (45)$$

with

$$\begin{aligned} C_{F_y}^i &= C_D^i = 1 - T(\alpha) \\ C_{F_y}^k &= C_D^k = R(\alpha) \cdot \cos 2\alpha \\ C_{F_z}^k &= C_L^k = R(\alpha) \cdot \sin 2\alpha . \end{aligned} \quad (46)$$

The course of the solar sail is determined by the lift/drag ratio which gives the course angle $\alpha_c$ :

For opaque materials

$$\tan \alpha_c = \frac{C_L^k}{C_D^i + C_D^k} = \frac{R(\alpha) \cdot \sin 2\alpha}{1 + R(\alpha) \cos 2\alpha} \quad (47)$$

For transparent materials without absorption

$$T(\alpha) + R(\alpha) = 1$$

$$\tan \alpha_c = \frac{\sin 2\alpha}{1 + \cos 2\alpha} = \tan \alpha \qquad (48)$$

$$\alpha_c = \alpha .$$

For an opaque plane sail, the values of the radiation force coefficients $C_L$ and $C_D$ as functions of R ($\alpha$) are given in Figure 9. The maximum of the lift $F_z$ is found by differentiation with respect to $\alpha$ of the expression:

$$F_z = p_0 \cdot A \cdot R \cdot \sin 2\alpha \cos \alpha$$

while R is kept constant. The maximum of $F_z$ occurs as the angle

$$\alpha = 35.28°$$

without dependence on the value of the reflectivity.

It is worth mentioning that, in the case of a transparent plane-parallel plate, we get a torque due to the parallel displacement of the light beams.

## E. The Spherical Surface

We consider a spherical satellite, the right hemisphere of which is illuminated by the sun's light (Figure 10). The radiation incident on an arbitrary annular section $dA_0$ is uniform over the circumference of the section. We get this ring-shaped element when we consider a surface element $da_0$ which is given by:

$$da_0 = r d\alpha \cdot b d\beta = r^2 \sin \alpha \, d\alpha \cdot d\beta .$$

By integration we have:

$$dA_0 = \int_{\beta=0}^{2\pi} da_0 = \int_{\beta=0}^{2\pi} r^2 \sin \alpha \, d\alpha \, d\beta \qquad (49)$$

$$= 2 r^2 \pi \sin \alpha \, d\alpha \equiv G(\alpha) \cdot d\alpha$$

The radiation force acting on a spherical calotte with the half-opening angle $\alpha_\nu$ is obtained by using the equations (35) through (40):

$$F = F_Y = F_Y^i + F_Y^r = p_0 \cdot A \cdot \left( C_{F_Y}^i + C_{F_Y}^r \right)$$

$$F_Y^i = p_0\, r^2 \pi \cdot 2 \sum_{t=1}^{\nu} \int_{\alpha_{t-1}}^{\alpha_t} (1 - T_t) \sin\alpha \cos\alpha \, d\alpha \qquad (50)$$

$$F_Y^r = p_0 \cdot r^2 \pi \cdot 2 \sum_{r=1}^{\nu} \int_{\alpha_{r-1}}^{\alpha_r} R_r \cdot \sin\alpha \cos 2\alpha \cos\alpha \, d\alpha$$

The radiation force coefficients $C_{F_Y}^i$ and $C_{F_Y}^r$ are found as

$$C_{F_Y}^i = \frac{2}{\sin^2\alpha_\nu} \sum_{t=1}^{\nu} \int_{\alpha_{t-1}}^{\alpha_t} (1 - T_t) \sin\alpha \cos\alpha \, d\alpha \qquad (51)$$

$$C_{F_Y}^r = \frac{2}{\sin^2\alpha_\nu} \sum_{r=1}^{\nu} \int_{\alpha_{r-1}}^{\alpha_r} R_r \cdot \sin\alpha \cos 2\alpha \cos\alpha \, d\alpha \qquad (52)$$

Here the functions $T_t$ and $R_r$ are given by the equations (26) through (29), and (36a, b). In the following, we consider opaque materials and then have $T_t = 0$. Of special interest are materials:

1. With total absorption, R = 0
2. With ideal reflection, R = 1. We obtain for the acting forces:

$$1.\quad F = F_Y^i = p_0 \cdot A \cdot C_{F_Y}^i$$

$$= p_0 \left( r \sin\alpha_\nu \right)^2 \pi \cdot C_{F_Y}^i \qquad (53)$$

$$C_{F_Y}^i = \frac{2}{\sin^2\alpha_\nu} \int_0^{\alpha_\nu} \sin\alpha \cos\alpha \, d\alpha = 1$$

$$2. \quad F = F_Y^i + F_Y^r$$

$$F_Y^r = p_0 \cdot A \cdot C^r_{F_{\alpha_\nu}} = p_0 \left(r \sin \alpha_\nu\right)^2 \pi \cdot C^r_{F_Y}$$

$$C^r_{F_Y} = \frac{2}{\sin^2 \alpha_\nu} \int_0 \sin \alpha \cos 2\alpha \cos \alpha \, d\alpha \qquad (54)$$

$$= \frac{2}{\sin^2 \alpha_\nu} \left[ - \frac{\cos^4 \alpha}{4} - \frac{\sin^2 \alpha}{4} \right]_0^{\alpha_\nu}$$

$$= \frac{1}{2 \sin^2 \alpha_\nu} \left( - \cos^4 \alpha_\nu - \sin^4 \alpha_\nu + 1 \right) = \cos^2 \alpha_\nu$$

$$F = p_0 \cdot A \cdot C_{F_Y} = p_0 A \left( 1 + \cos^2 \alpha_\nu \right) \qquad (55)$$

In Figure 11 the values for the radiation force coefficient in the above two cases are plotted in their dependency on the half-opening angle $\alpha_\nu$. In the case of a hemisphere with ideal reflection, $C^r_{F_Y}$ becomes zero and the effect of the radiation is equal to that of a hemisphere with total absorption. But this result is true only with the assumption $R_r = 1$. In the interval $0^\circ \leqq \alpha_\nu \leqq 45^\circ$, spherical surfaces with a reflectivity $0 < R < 1$ will be characterized by curves for the radiation force coefficient between curves 1 and 2 of Figure 11. Values for $C_{FY}$ for spherical surfaces with larger opening angle, $\frac{\pi}{4} < \alpha \leqq \frac{\pi}{2}$, are not limited by these two lines.

The values for the reflectivity, R (n $(\lambda), \alpha$), are not available today. Therefore, we are unable to calculate the force acting on a metallic sphere. But we can calculate the limits for all kinds of materials. The upper limit will be obtained with a surface material excluding any force components opposite to the light direction. The lowest values of $C_{F_Y}$ $(\alpha)$ appear when the component of the force $F^r$ in direction of the incident light is zero and when, in addition, the component opposite to it is acting in its full amount. The limiting values of $C_{FY}$ are given by the curves 3 and 4 in Figure 11. On the body

which is represented by curve 3, the area of the surface between the angles of incidence $\alpha = 0^o$ and $\alpha = 45^o$ consists of an ideally reflecting material while the zone from $\alpha = 45^o$ to $\alpha = 90^o$ is assumed as totally absorbing. We get the minimum force if the inner part of the sphere's surface has total absorption and the outer zone possesses ideal reflection. When the entire hemisphere is illuminated, the values of $C_{F_Y}$ are limited by:

$$0.75 \leqq C_{F_Y} \leqq 1.25$$

F. The Cylindrical Surface

We consider 1) a right circular cylinder, with its axis of symmetry in the z-direction, as indicated in Figure 12a, and 2) a cylindrical surface, which might be part of an arbitrary body (Figure 12b). For symmetry reasons the resultant of the force components in x- and z-directions becomes zero. The y-component can be calculated from the equations (21) and (25). The angle $\alpha$ lies in the x-y-plane. In the case of Figure 12a we have $\alpha_\nu = \frac{\pi}{2}$. The surface element $dA_o$ is:

$$dA_o = r \cdot l \cdot d\alpha \tag{56}$$

If $T_t = 0$, the force of the incoming light becomes:

$$\begin{aligned} F_y^i &= 2rl \cdot p_o \cdot \int_0^{\alpha_\nu} \cos\alpha\, d\alpha \\ &= 2rl \sin\alpha_\nu \cdot p_o \cdot \frac{1}{\sin\alpha_\nu} \int_0^{\alpha_\nu} \cos\alpha\, d\alpha \end{aligned} \tag{57}$$

$$C_{F_y}^i = \frac{1}{\sin\alpha_\nu} \int_0^{\alpha_o} \cos\alpha_\nu\, d\alpha = 1 \tag{58}$$

The reflected light contributes the force:

$$\begin{aligned} F_y^r &= 2rl\, p_o \sum_{\rho=1}^{\nu} \int_{\alpha_{\rho-1}}^{\alpha_\rho} R_\rho \cos 2\alpha \cos\alpha\, d\alpha \\ &= p_o \cdot A \cdot C_{F_y}^r . \end{aligned} \tag{59}$$

$$C_{F_Y}^{\mu} = \frac{1}{\sin\alpha_\nu} \sum_{\mu=1}^{\nu} \int_{\alpha_{\mu-1}}^{\alpha_\mu} R_\mu \cdot \cos 2\alpha \cos\alpha \, d\alpha \tag{60}$$

$$= \frac{1}{\sin\alpha_\nu} \sum_{\mu=1}^{\nu} R_\mu \left[ \frac{\sin\alpha}{2} + \frac{\sin 3\alpha}{6} \right]_{\alpha_{\mu-1}}^{\alpha_\mu}$$

In Figure 13, curves 1 and 2 give $C_{F_Y}$ for $R_r = 0$ and $R_r = 1$. With the same arrangement as used with the spherical surface, we get the limits for a cylindrical body. Curve 3 in Figure 13 represents the case where $R_r = 0$ from $\alpha = 0^\circ$ to $\alpha = 45^\circ$ and where $R_r = 1$ from $\alpha = 45^\circ$ to $\alpha = 90^\circ$. Curve 4 shows the opposite case with respect to the reflection. For the whole cylinder we get the limits:

$$0.862 \leqq C_{F_Y} \leqq 1.471.$$

G. The Conical Surface

Here we deal with two cases only: (a) where the geometrical axis is perpendicular to the sun's radiation, and (b) where the cone's vertex points towards the sun. (Figures 14a and b.)

a. Cone Axis Perpendicular to the Sun's Radiation

The surface element $dA_o$ at point P is a rectangle formed by the slant height ds and by $r \cdot d\chi$, where r is the radius of the circle through the point P. (Figure 14c.) The slant height is counted from the vertex; thus we have $r = s \cdot \sin\omega$, and the surface element $dA_o$ becomes:

$$dA_o = ds \cdot r\,d\chi = s \cdot \sin\omega \, ds \cdot d\chi \tag{61}$$

From Figure 14a we calculate the exterior normal $\bar{n}$:

$$\bar{n} = \cos\omega \sin\chi \, \bar{i} + \cos\omega \cos\chi \, \bar{j} + \sin\omega \, \bar{k} \tag{62}$$

for which we had previously obtained the general relation:

$$\bar{n} = \sin\gamma \sin\alpha \, \bar{i} + \cos\alpha \, \bar{j} + \cos\gamma \sin\alpha \, \bar{k} \tag{24}$$

By comparison of the coefficients we get:

$$\sin \gamma = \frac{\cos \omega \cdot \sin X}{\sin \alpha}$$

$$\cos \gamma = \frac{\sin \omega}{\sin \alpha} \tag{63}$$

$$\cos \alpha = \cos \omega \cdot \cos X .$$

In the case of spherical surfaces as well as for hyperbolic and parabolic forms, it is practical to integrate the force produced by the radiation over the incident angle of the light. Here the third of the equations (63) shows us that $\alpha$ is a function of the half opening angle $\omega$ and of the angle $X$. When a conical surface is only a part of a circular cone as in Figure 14, the size and the shape are determined by a suitably chosen value of the angle $X$. Now we insert the results (63) into the equations (21) and (25). The differential of the radiation force for the incoming light is then found as:

$$d\bar{F}^{i} = -p_0 \sin\omega \cos\omega \left(1 - T(\alpha)\right) \cos X \cdot s\, ds\, dX\, \bar{j} , \tag{64}$$

where the angle $X$ plays the role formerly assigned to the angle $\alpha$. Similarly, for the reflected light:

$$\begin{aligned} d\bar{F}^{r} = & -p_0 \cdot \sin\omega \cos\omega\, R(\alpha) \cos X \cdot s\, ds\, dX \cdot \\ & \cdot \Big[ 2 \cos^2\omega \sin X \cos X\, \bar{i} + \\ & \quad (2 \cos^2\omega \cos^2 X - 1)\, \bar{j} + \\ & \quad 2 \sin\omega \cos\omega \cos X\, \bar{k} \Big] \end{aligned} \tag{65}$$

Formal integration of the differential equation 64 yields:

$$\bar{F}^{i} = -p_o \sin\omega\cos\omega \int_{s=0}^{s^*} s \sum_{t=1}^{\nu} 2\int_{\chi_{t-1}}^{\chi_t} (1-T_t)\cos\chi\, d\chi\, ds\, \bar{j}$$

$$= -p_o \cdot \frac{s^{*2}}{2} \cdot \sin\omega\cos\omega \sum_{t=1}^{\nu} 2\int_{\chi_{t-1}}^{\chi_t} (1-T_t)\cdot\cos\chi\, d\chi\, \bar{j} \qquad (66)$$

with

$\chi_o = 0$ and $\chi_t = \arccos\left(\frac{\cos\alpha_t}{\cos\omega}\right)$,

where $\alpha_t$ is in the interval $\frac{\pi}{2} \geqq \alpha_t \geqq \omega$.

From Figure 14d, we can see that the projected area of a cone is $s^{*2}\sin\omega\cos\omega$ ; that of the cone segment considered then becomes:

$$A = s^{*2}\cdot\sin\omega\cdot\cos\omega\cdot\sin\chi_\nu .$$

Putting the last expression into equation (66)

$$\bar{F}^{i} = -p_o\cdot A\cdot\frac{1}{\sin\chi_\nu}\cdot\sum_{t=1}^{\nu}\int_{\chi_{t-1}}^{\chi_t}(1-T_t)\cos\chi\, d\chi\, \bar{j} \qquad (67)$$

From equation 65, it can be seen that force elements acting on surface elements, which are symmetric with respect to the y-z-plane, cancel each other, so that the i-component of the integrated force in zero. As a consequence:

$$\bar{F}^{k} = -p_o\cdot A\cdot\frac{1}{\sin\chi_\nu}\cdot\sum_{k=1}^{\nu}\int_{\chi_{k-1}}^{\chi_k} R_k \cdot$$

$$\left[(2\cos^2\omega\cos^3\chi - \cos\chi)\bar{j} + + 2\sin\omega\cos\omega\cos^2\chi\,\bar{k}\right]. \qquad (68)$$

From these equations, we get the following expressions for the radiation force coefficients:

$$C_{F_y}^{i} = \frac{1}{\sin \chi_\nu} \sum_{t=1}^{\nu} (1-T_t)\left[\sin \chi\right]_{\chi_{t-1}}^{\chi_t}$$

$$C_{F_x}^{r} = 0$$

$$C_{F_y}^{r} = \frac{1}{\sin \chi_\nu} \cdot \sum_{r=1}^{\nu} R_r \left[\frac{2}{3}\cos^2\omega \sin \chi \left(\cos^2\chi + 2\right) - \sin \chi\right]_{\chi_{r-1}}^{\chi_r} \tag{69}$$

$$C_{F_z}^{r} = \frac{2 \sin\omega \cos\omega}{\sin \chi_\nu} \sum_{r=1}^{\nu} R_r \left[\frac{\sin 2\chi}{4} - \frac{\chi}{2}\right]_{\chi_{r-1}}^{\chi_r}$$

With an ideally reflecting full cone the coefficients become:

$$C_{F_y}^{i} = 1$$

$$C_{F_y}^{r} = \frac{4}{3}\cos^2\omega - 1 \tag{70}$$

$$C_{F_z}^{r} = \frac{\pi}{2} \cdot \sin\omega \cdot \cos\omega$$

The total radiation force then is:

$$\bar{F} = -p_0 \cdot A\left(\frac{4\cos^2\omega}{3}\,\bar{j} + \frac{\pi}{2} \cdot \sin\omega \cos\omega\,\bar{k}\right) \tag{71}$$

These forces move the cone obliquely away from the light source. The course angle $\alpha_c$ in the ( -j, - k) - plane is given by

$$\tan \alpha_c = \frac{3\pi}{8} \tan \omega \tag{72}$$

b. <u>Cone's Axis Pointing Toward the Sun</u>

The radiation illuminates the lateral area $A_0$ (Figure 14b)

$$A_0 = r \cdot \pi \cdot s = \frac{r^2 \pi}{\cos \alpha} \tag{73}$$

The half-opening angle $\omega$ and the angle of incidence $\alpha$ are connected by:

$$\alpha = 90° - \omega$$

If $T_t = 0$, the force of the incoming radiation becomes

$$\bar{F}^i = -p_0 \frac{r^2 \pi}{\cos \alpha} \cdot \cos \alpha \, \bar{j} = -p_0 r^2 \pi \bar{j} = -p_0 \cdot A \cdot C_{F_Y}^i \bar{j} \tag{74}$$

where

$$C_{F_Y}^i = 1$$

For symmetry reasons, the reflected light has only a force component in the j-direction, and we have:

$$F_Y^r = -p_0 r^2 \pi \cdot R(\alpha) \cos 2\alpha \, \bar{j} \tag{75}$$

with

$$C_{F_Y} = 1 + \cos 2\alpha = 2 \cos^2 \alpha .$$

In the case of ideal reflection, R ($\alpha$) = 1,

$$C_{F_Y} = 1 + \cos 2\alpha = 2 \cos^2 \alpha . \tag{76}$$

The limits are:

$$0 < C_{F_Y} < 2$$

Long tapers, due to the reflection force component opposite to the light beam, have a small resultant force component in light direction. As an example, for a slender taper with s/r = 5, $\alpha$ becomes $\approx 78.5^o$, and $C_{FY} \approx 0.08$, if $R \approx 1$ (highly reflecting surface). When the cone angle $\alpha$ is larger than $45^o$, $C_{FY}$ becomes smaller than unity.

### H. The Parabolic Surface

We consider a parabolic body as sketched in Figure 15a. As in the case of the sphere, we select an arbitrary annular section $dA_o$ and a surface element $da_o$ on it. Let p be the half-parameter of the parabola. The shape then is given by p, and the size is determined by the half-opening angle $\alpha_\nu$.

For the derivation of the pertinent formulas, we consider Figure 15b. The equation of the parabola has the form:

$$z^2 = -2p\cdot y, \tag{77}$$

and as

$$z = p\cdot \tan\alpha \tag{78}$$

the radius of curvature of $dA_o$ becomes:

$$|\rho| = \left|\frac{(z'^2+1)^{3/2}}{z''}\right| = \left|p\cdot\frac{1}{\cos^3\alpha}\right| \tag{79}$$

Thus the surface element

$$da_o = \rho\, d\alpha\cdot z\, d\beta = p^2\cdot\frac{\sin\alpha}{\cos^4\alpha}\, d\alpha\, d\beta. \tag{80}$$

The ring-shaped element $dA_o$ is then obtained as:

$$dA_o = 2\pi p^2\frac{\sin\alpha}{\cos^4\alpha}\, d\alpha = G(\alpha)\, d\alpha \tag{81}$$

We exclude the infinite parabola so that $\alpha$ remains smaller than $\frac{\pi}{2}$.

By using the equations (35) through (40), the radiation forces are found as:

$$F_y^i = 2p^2\pi \cdot p_0 \sum_{t=1}^{\nu} \int_{\alpha_{t-1}}^{\alpha_t} (1-T_t)\frac{\sin\alpha}{\cos^3\alpha}\,d\alpha$$

$$= p^2\pi \cdot p_0 \sum_{t=1}^{\nu} (1-T_t)\left[-\frac{1}{\cos^2\alpha}\right]_{\alpha_{t-1}}^{\alpha_t} \tag{82}$$

$$F_y^r = 2p^2\pi \cdot p_0 \sum_{r=1}^{\nu} \int_{\alpha_{r-1}}^{\alpha_r} R_r \cdot \frac{\sin\alpha\cos 2\alpha}{\cos^3\alpha}\,d\alpha$$

$$= 2p^2\pi \cdot p_0 \sum_{r=1}^{\nu} \int_{\alpha_{r-1}}^{\alpha_r} R_r\left(2\tan\alpha - \frac{\sin\alpha}{\cos^3\alpha}\right)d\alpha \tag{83}$$

$$= p^2\pi \cdot p_0 \sum_{r=1}^{\nu} R_r\left[-4\ln\cos\alpha + \frac{1}{\cos^2\alpha}\right]_{\alpha_{r-1}}^{\alpha_r}.$$

The projected area A of the parabolic body is given by:

$$A = z^2\pi = p^2\tan^2\alpha_\nu \cdot \pi \tag{84}$$

Thus we have for the radiation force coefficients:

$$C_{F_y}^i = \frac{1}{\tan^2\alpha_\nu} \cdot \sum_{t=1}^{\nu} (1-T_t)\cdot\left[-\frac{1}{\cos^2\alpha}\right]_{\alpha_{t-1}}^{\alpha_t}$$

$$C_{F_y}^r = \frac{1}{\tan^2\alpha_\nu} \cdot \sum_{r=1}^{\nu} R_r \cdot \left[-4\ln\cos\alpha + \frac{1}{\cos^2\alpha}\right]_{\alpha_{r-1}}^{\alpha_r} \tag{85}$$

We determine $C_{F_y}$ for different cases of R and T:

<u>Opaque Material</u>

1. Material with total absorption

$$T = 0, \quad R = 0$$

$$C_{F_y} = 1 \tag{86}$$

2. Medium with ideal reflection:

$$T = 0, \quad R = 1$$

$$C_{F_y} = \frac{4}{\tan^2\alpha_\nu}\left[-\ln\cos\alpha_\nu\right] \tag{87}$$

3. Medium with constant reflectivity:

$$T = 0, \quad 0 < R < 1$$

$$C_{F_y} = \frac{1}{\tan^2\alpha_\nu}\left[-4R\ln\cos\alpha + \frac{R-1}{\cos^2\alpha}\right]_{\alpha_{k-1}}^{\alpha_k} \tag{88}$$

Transparent Material

4. Medium without absorption

R + T = 1

$$C_{F_y} = \frac{4}{\tan^2\alpha_\nu}\sum_{k=1}^{\nu} R_k\left[-\ln\cos\alpha\right]_{\alpha_{k-1}}^{\alpha_k} \tag{89}$$

From equation (87) it can be seen that in the present case $C_{F_y}$ is bounded by:

$$0 < C_{F_y} < 2.$$

## III. CONCLUDING REMARKS

These studies were made for application to computations of both a satellite's attitude and trajectory, which are to take into account the effects of the radiation pressure.

The radiation force acting on a body is a function of certain physical properties, of which the reflectitivy of the body's surface is the most important. The force F can be expressed in the form:

$$F = p_o \times A \times C_F$$

"A" means the projected area of the body and $p_o$ is the radiation pressure in the vicinity of the earth. The function $C_F$ was introduced in analogy to the aerodynamic expression for the drag, and may be called the "radiation force coefficient." The purpose of this report was to derive accurate formulas and limits for the radiation force coefficient for certain convex bodies. For finite bodies, the result is that $C_F$ is limited by:

$$0 < C_F \leqq 2$$

Plane Surfaces

$$0 < C_F \leqq 2$$

Spheres

$$0.75 \leqq C_F \leqq 1.25$$

Cylinders
(Illuminated perpendicularly to the axis of symmetry.)

$$0.862 \leqq C_F \leqq 1.471$$

Cones in Special Positions
(See Figures 14a and 14b)

$$0 < C_F < 2$$

Parabolic Bodies

$$0 < C_F < 2$$

## IV. APPENDIX

The total-radiation, E, of a black body, coming from 1 $cm^2$ of the surface, is given by Stefan-Boltzmann's law, which states that the total rate of radiation varies as the fourth power of the absolute temperature T, that is:

$$E = \sigma \cdot T^4.$$

T is the absolute temperature, and $\sigma$ a constant.

The rate of emission, $E_\lambda$, of the radiation of any particular wave-length $\lambda$, which passes in 1 second through each cross section of a radiation cone of the aperture $\Omega$ = 1, is given by Planck's equation:

$$E_\lambda \, d\lambda = c_1 \lambda^{-5} \left( e^{c_2/\lambda T} - 1 \right)^{-1} d\lambda$$

$$= c^2 h \, \lambda^{-5} \left( e^{ch/k\lambda T} - 1 \right)^{-1} d\lambda.$$

Here the notations are:

$$c_1 = c^2 h$$

$$c_2 = ch/k$$

$$\sigma = \frac{2\pi^5}{15} \cdot \frac{c_1}{c_2^4} = \frac{2\pi^5 k^4}{15 c^2 h^3} \left[ \frac{\text{erg}}{\text{cm}^2 \, \text{degree}^4 \, \text{sec}} \right]$$

c = the velocity of light in vacuo

k = the gas constant for 1 molecule (Boltzmann's constant)

h = Planck's quantum of action; with the values:

c = 299 850 km $sec^{-1}$

k = 1.372 Watt sec $degree^{-1}$

h = 0.655 x $10^{-33}$ Watt $sec^2$

$$c_1 = 5.9 \times 10^{-6} \text{ erg cm}^2 \text{ sec}^{-1} = 5.9 \times 10^{-13} \text{ Watt cm}^2$$

$$= 1.42 \times 10^{-13} \text{ cal cm}^2 \text{ sec}^{-1}$$

$$c_2 = 1.43 \text{ cm degree}$$

$$\sigma = 5.8 \times 10^{-5} \cdot \frac{\text{erg}}{\text{cm}^2 \text{ sec degree}^4} = 5.8 \times 10^{-12} \frac{\text{Watt}}{\text{cm}^2 \text{ degree}^4}$$

$$= 1.39 \times 10^{-12} \frac{\text{cal}}{\text{cm}^2 \text{ sec degree}^4}$$

The connection between Planck's and Stefan-Boltzmann's equation is given by integration of Planck's equation over the whole spectrum, as follows:

$$\int_0^\infty E_\lambda \, d\lambda = c_1 \int_0^\infty \frac{\lambda^{-5}}{e^{c_2/\lambda T} - 1} \, d\lambda = \frac{\sigma}{2\pi} T^4$$

This is the energy within the solid angle $\Omega = 1$. (Reference 8).

TABLE I

THE DISTRIBUTION OF ENERGY IN THE SPECTRUM OF THE SUN

Extraterrestical Radiation Intensity of the Sun $i_{o\lambda}$ [in $10^{-3}$ cal $cm^{-2}$ $min^{-1}$] taken from M. Nicolet, Archiv Meteor., Geophys., Bioklim. B 3 (1951) 209 for $\Delta\lambda$ = 100 AE. Values in Smithsonian Scale 1913 minus 2.4%, Solar Constant = 1.98 cal $cm^{-2}$ $min^{-1}$.

| $\lambda[\mu]$ | $i_{o\lambda}$ | $\lambda[\mu]$ | $i_{o\lambda}$ | $\lambda[\mu]$ | $i_{o\lambda}$ | $\lambda[\mu]$ | $i_{o\lambda}$ |
|---|---|---|---|---|---|---|---|
| 0.20 | - | 0.50 | 29.7 | 0.80 | 16.5 | 1.50 | 4.3 |
| 0.21 | - | 0.51 | 29.7 | 0.81 | 16.3 | 1.55 | 3.9 |
| 0.22 | 0.2 | 0.52 | 27.7 | 0.82 | 15.9 | 1.60 | 3.7 |
| 0.23 | 0.5 | 0.53 | 28.9 | 0.83 | 15.6 | 1.65 | 3.2 |
| 0.24 | 0.6 | 0.54 | 28.9 | 0.84 | 15.3 | 1.70 | 3.0 |
| 0.25 | 0.8 | 0.55 | 27.6 | 0.85 | 15.0 | 1.75 | 2.7 |
| 0.26 | 1.9 | 0.56 | 28.8 | 0.86 | 14.7 | 1.80 | 2.4 |
| 0.27 | 2.4 | 0.57 | 28.4 | 0.87 | 14.4 | 1.85 | 2.3 |
| 0.28 | 2.2 | 0.58 | 28.4 | 0.88 | 14.1 | 1.90 | 2.0 |
| 0.29 | 5.1 | 0.59 | 27.5 | 0.89 | 13.8 | 1.95 | 1.9 |
| 0.30 | 5.9 | 0.60 | 27.2 | 0.90 | 13.5 | 2.00 | 1.7 |
| 0.31 | 8.8 | 0.61 | 26.3 | 0.91 | 13.2 | 2.05 | 1.6 |
| 0.32 | 10.3 | 0.62 | 25.7 | 0.92 | 13.0 | 2.10 | 1.4 |
| 0.33 | 12.6 | 0.63 | 25.1 | 0.93 | 12.7 | 2.15 | 1.3 |
| 0.34 | 12.4 | 0.64 | 24.5 | 0.94 | 12.4 | 2.20 | 1.2 |
| 0.35 | 13.1 | 0.65 | 24.1 | 0.95 | 12.2 | 2.25 | 1.1 |
| 0.36 | 13.5 | 0.66 | 23.6 | 0.96 | 11.9 | 2.30 | 1.0 |
| 0.37 | 13.5 | 0.67 | 23.0 | 0.97 | 11.7 | 2.35 | 0.9 |
| 0.38 | 13.3 | 0.68 | 22.5 | 0.98 | 11.4 | 2.40 | 0.8 |
| 0.39 | 14.2 | 0.69 | 21.9 | 0.99 | 11.2 | 2.45 | 0.7 |
| 0.40 | 22.1 | 0.70 | 21.2 | 1.00 | 11.0 | 2.50 | 0.7 |
| 0.41 | 25.1 | 0.71 | 20.7 | 1.05 | 10.0 | 3.00 | 0.37 |
| 0.42 | 25.2 | 0.72 | 20.2 | 1.10 | 8.7 | 4.00 | 0.14 |
| 0.43 | 23.5 | 0.73 | 19.8 | 1.15 | 7.8 | 5.00 | 0.06 |
| 0.44 | 27.4 | 0.74 | 19.0 | 1.20 | 7.2 | 6.00 | 0.03 |
| 0.45 | 30.0 | 0.75 | 18.7 | 1.25 | 6.6 | 7.00 | 0.01 |
| 0.46 | 30.9 | 0.76 | 18.4 | 1.30 | 6.1 | | |
| 0.47 | 30.8 | 0.77 | 17.8 | 1.35 | 5.6 | | |
| 0.48 | 31.2 | 0.78 | 17.4 | 1.40 | 5.1 | | |
| 0.49 | 29.0 | 0.79 | 16.9 | 1.45 | 4.7 | | |

TABLE II

LIST OF ELEMENTS REPRESENTED IN FIGURE 4

| ELEMENT | SYMBOL | N = n - ki | $\lambda$ [Å] |
|---|---|---|---|
| Aluminum | Al | 0.78 - 2.85i | 4310 |
| Cesium | Cs | 0.36 - 0.86i | 4550 |
| Chromium | Cr | 1.64 - 3.69i | 2570 |
| Cobalt | Co | 1.10 - 1.43i | 2313 |
| Copper | Cu | 1.39 - 1.46i | 2313 |
| Gold | Au | 1.47 - 1.44i | 2540 |
| Iron | Fe | 1.01 - 0.88i | 2573 |
| Lead | Pb | 2.01 - 3.48i | 5890 |
| Magnesium | Mg | 0.37 - 4.42i | 5893 |
| Manganese | Mn | 0.66 - 1.19i | 2570 |
| Nickel | Ni | 0.87 - 1.24i | 2573 |
| Platinium | Pt | 1.29 - 1.96i | 2749 |
| Potassium | K | 0.08 - 1.00i | 4720 |
| Rhodium | Rn | 1.54 - 4.67i | 5790 |
| Silver | Ag | 1.41 - 1.11i | 2263 |
| Sodium | Na | 0.06 - 1.84i | 4350 |
| Tantalum | Ta | 2.10 - 2.18i | 4730 |
| Tin | Sn | 1.12 - 3.33i | 2570 |
| Vanadium | V | 2.55 - 3.08i | 4970 |
| Wolfram | W | 2.76 - 2.71i | 5790 |
| Zinc | Zn | 0.55 - 0.61i | 2573 |
| Steel | St | 1.47 - 2.03i | 2540 |

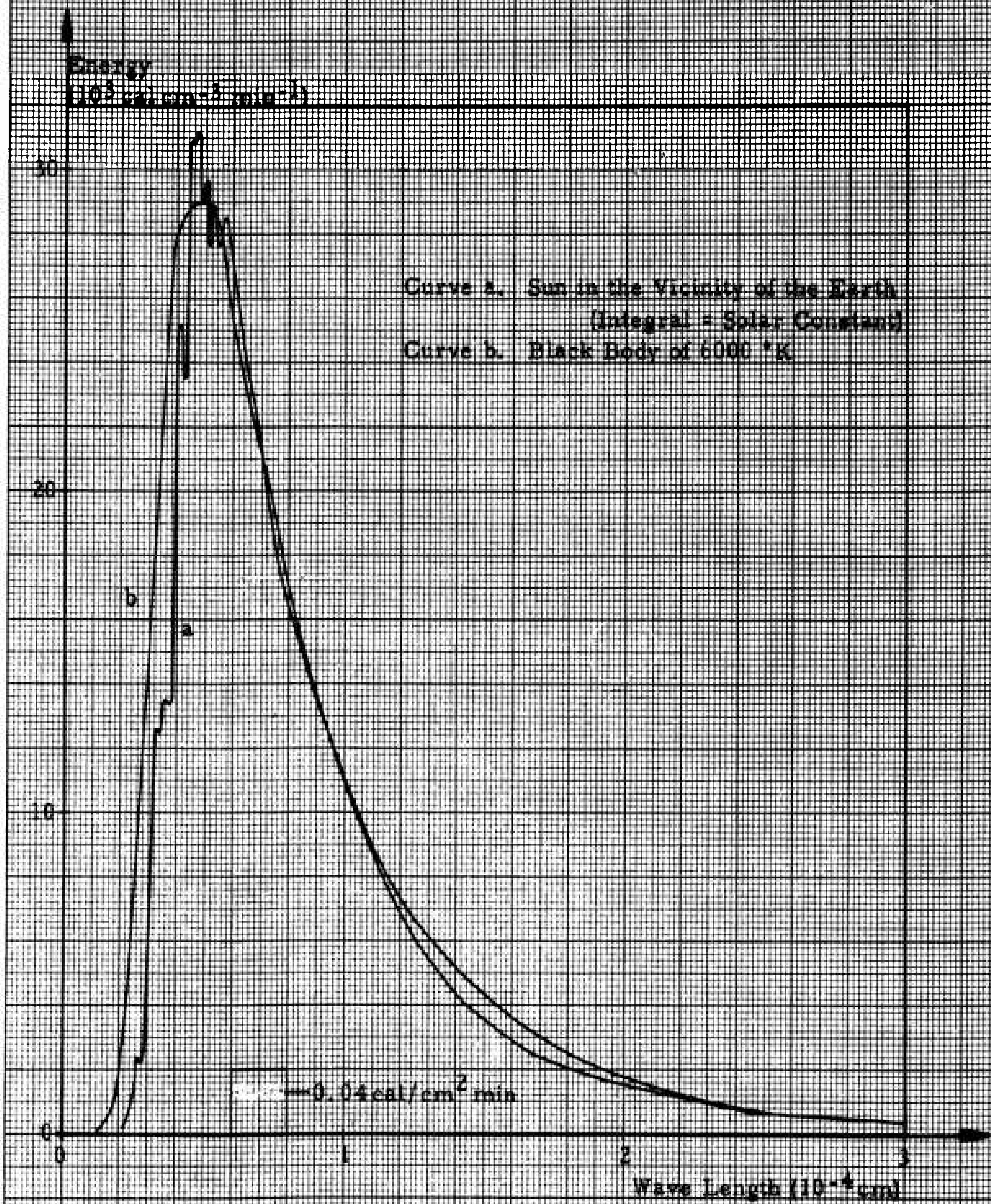



Figure 1 The Energy Distribution of the Spectrum

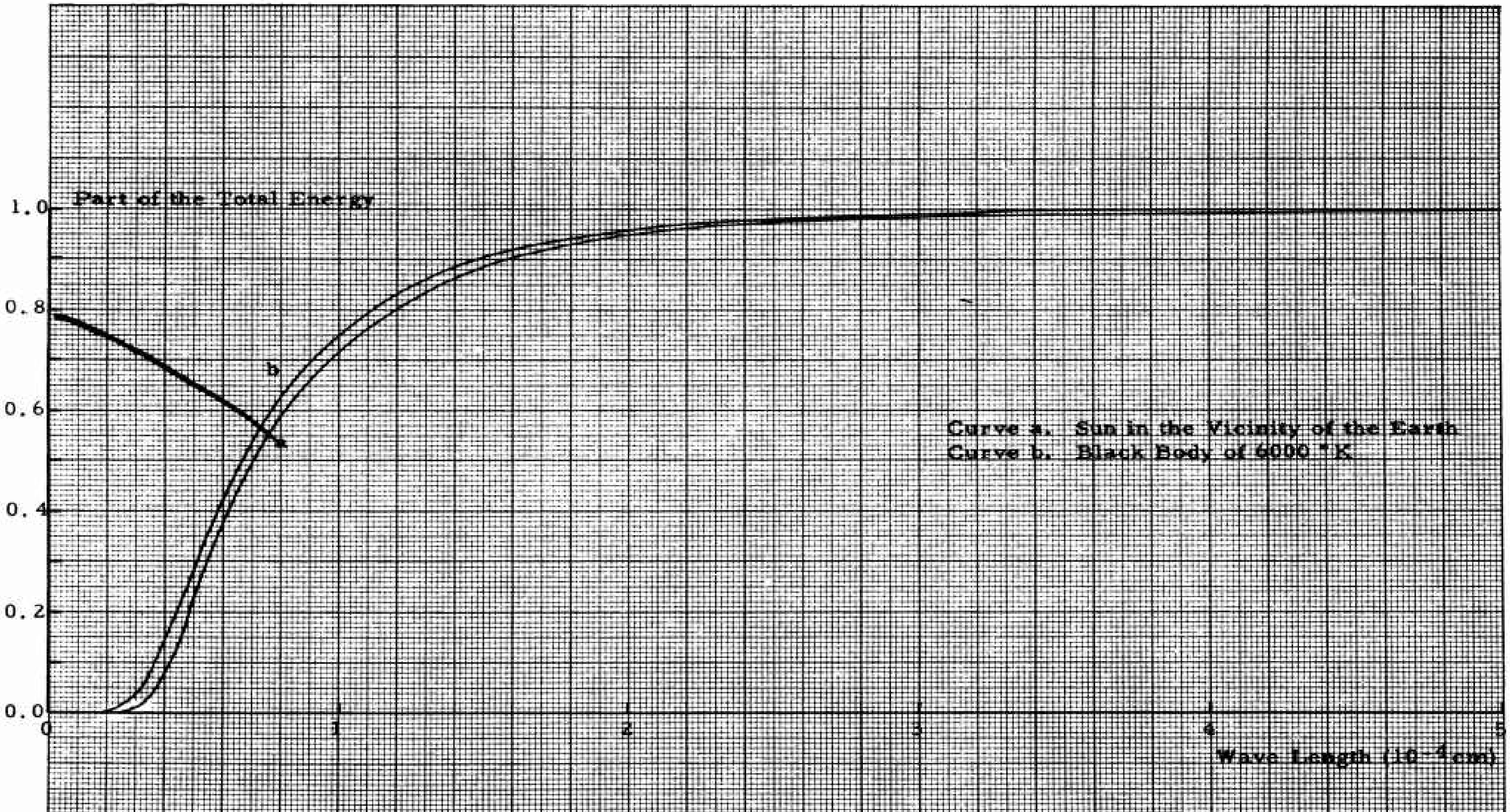



Figure 2 The Energy of the Spectrum Integrated from its Beginning in the Ultraviolet, in Parts of the Total Energy

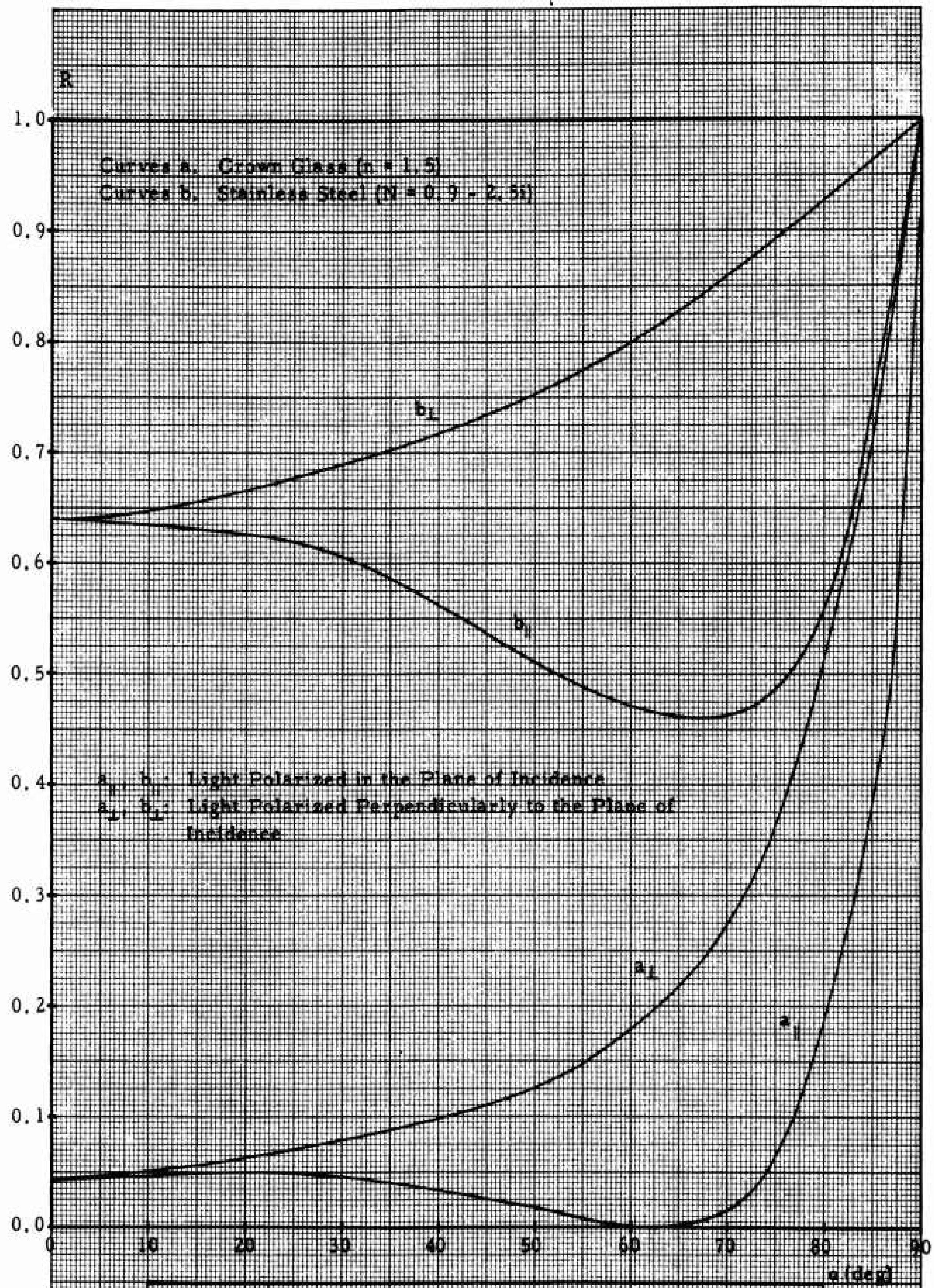
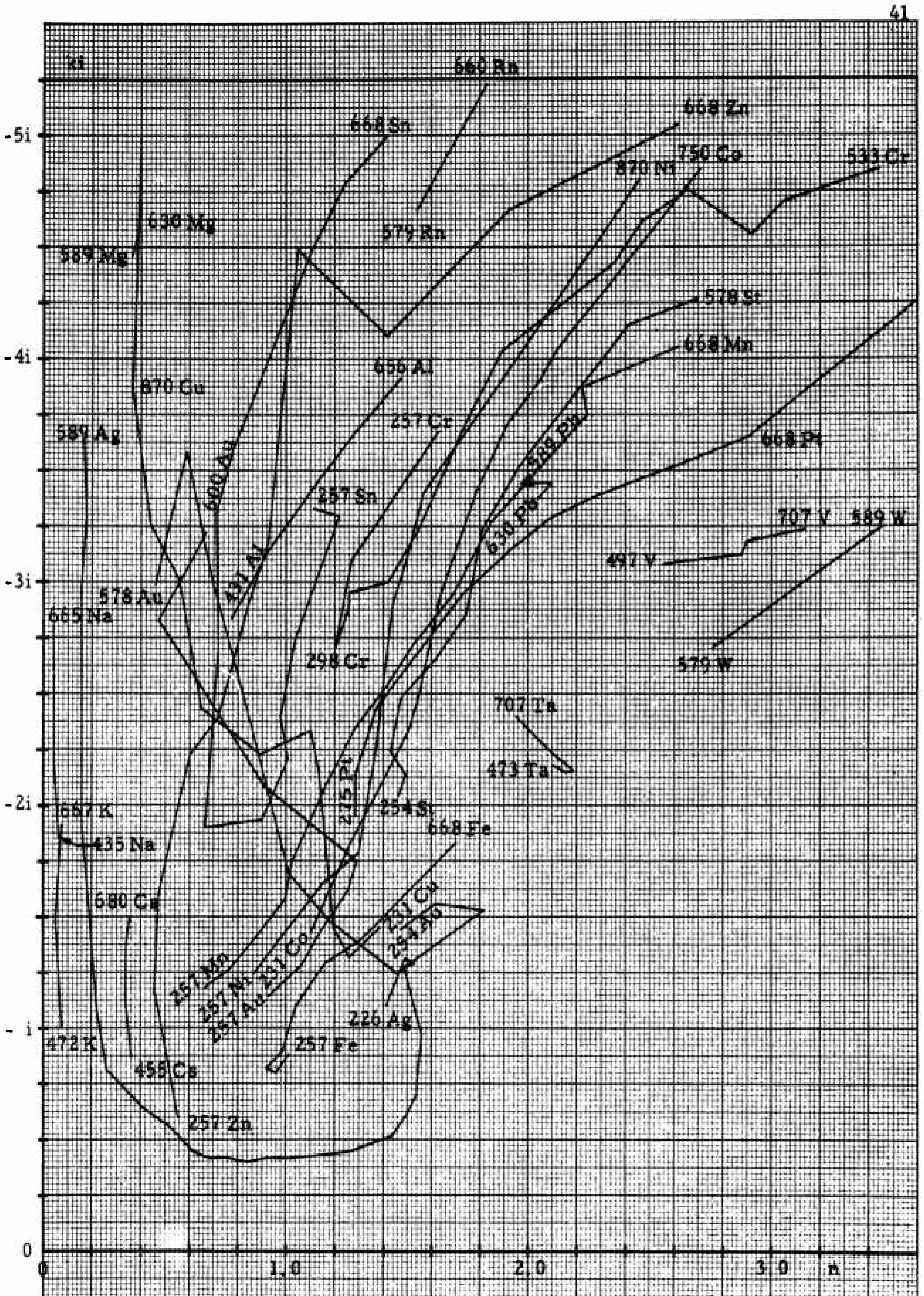



Figure 3 The Variation of Reflectivity with Angle of Incidence




| | |
|---|---|
| Figure 4 | The Gaussian Plane of the Refractive Indices, N = n - ki, with Characteristic Curves for a Number of Elements. (See Table II) |

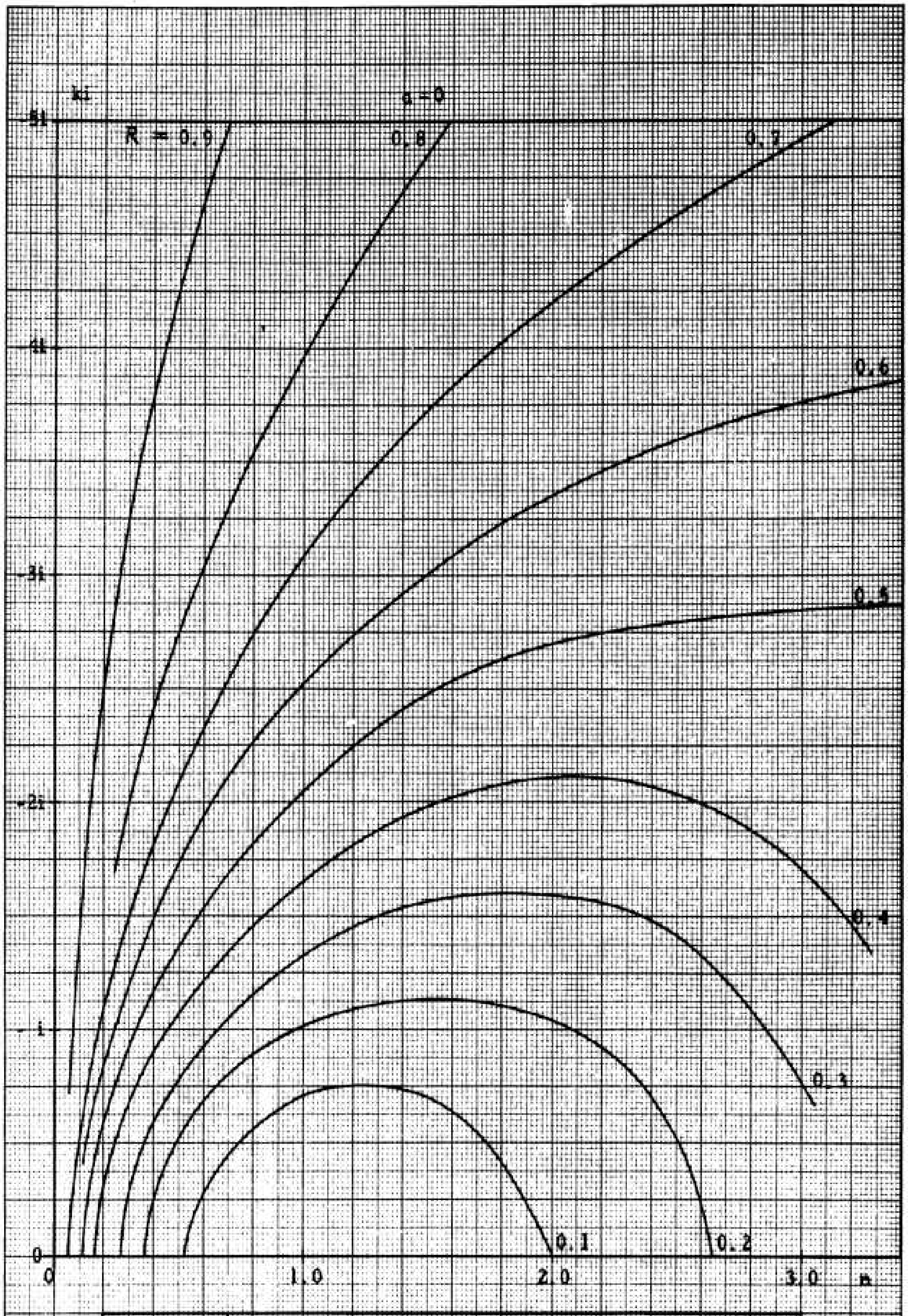
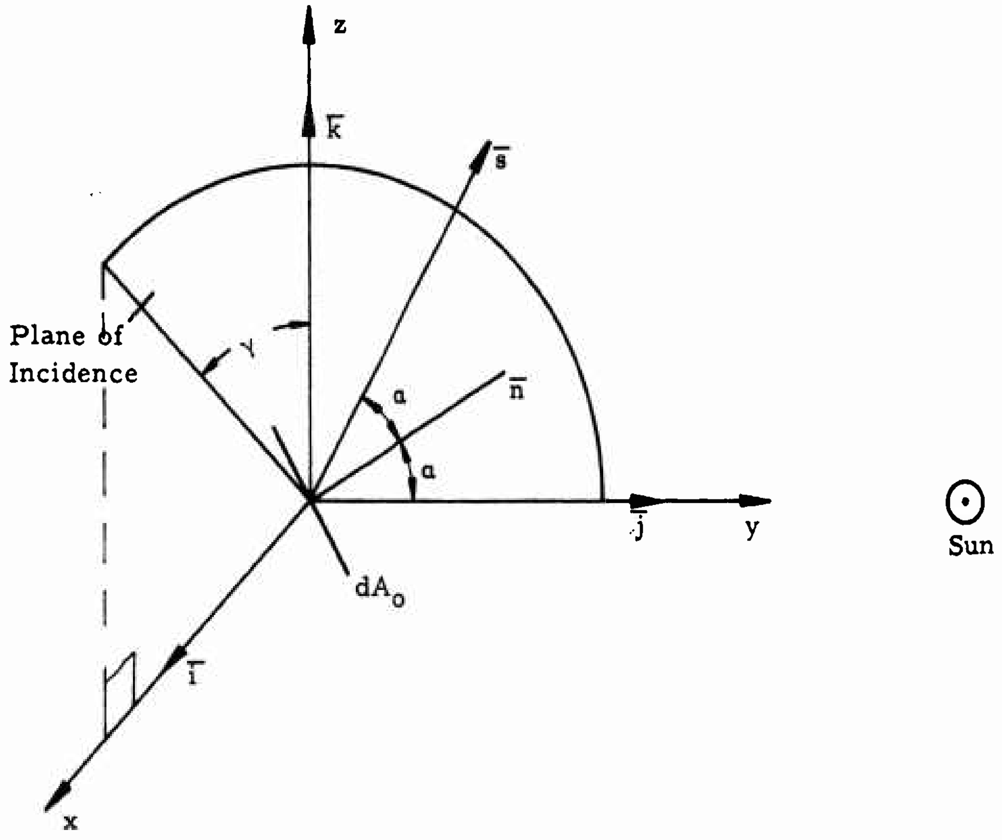
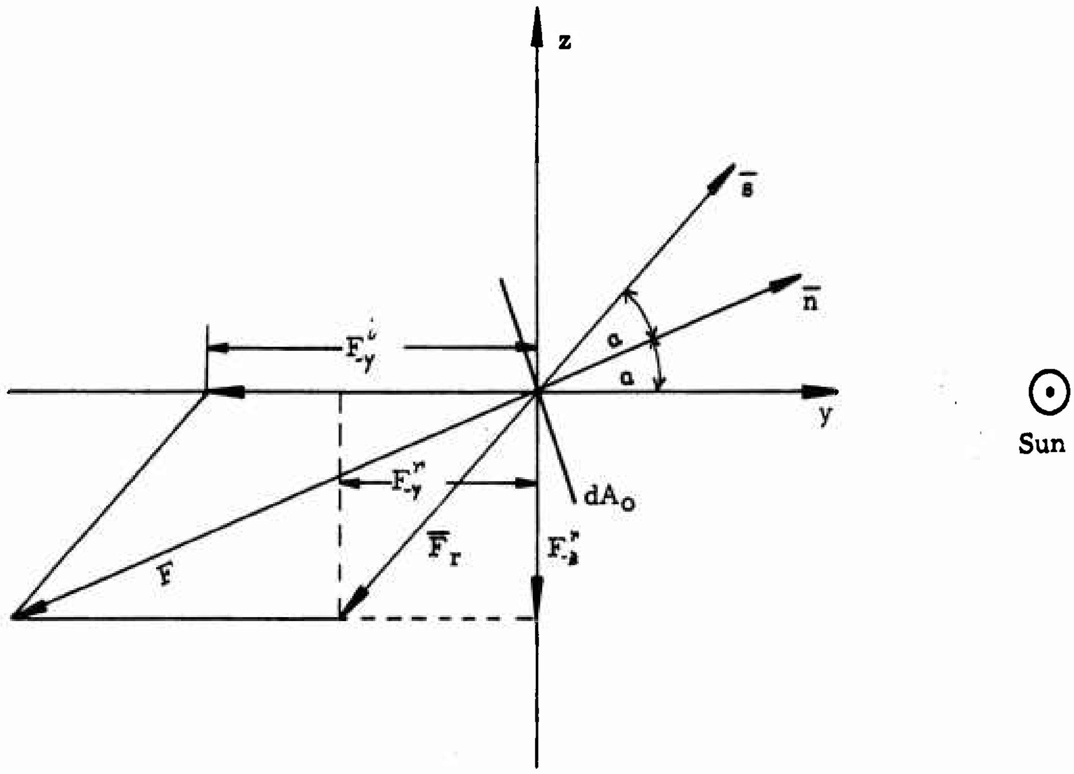



Figure 5

The Reflectivity of Light at Normal Incidence in Parts of the Incoming Intensity




| Figure 6 | The Surface Element $dA_o$ |
| --- | --- |




| Figure 7 | The Radiation Force Acting on a Surface Element $dA_o$ of a Symmetrical Body |
|---|---|

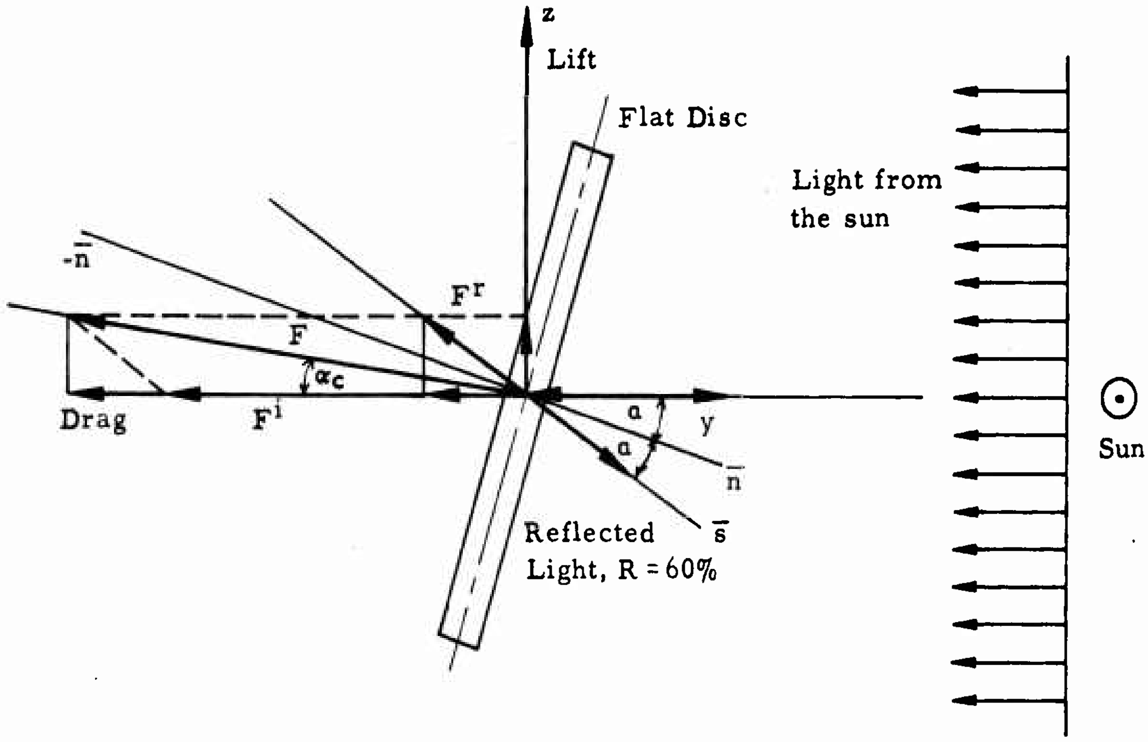



Figure 8 The Flight Characteristic of a Solar Sail

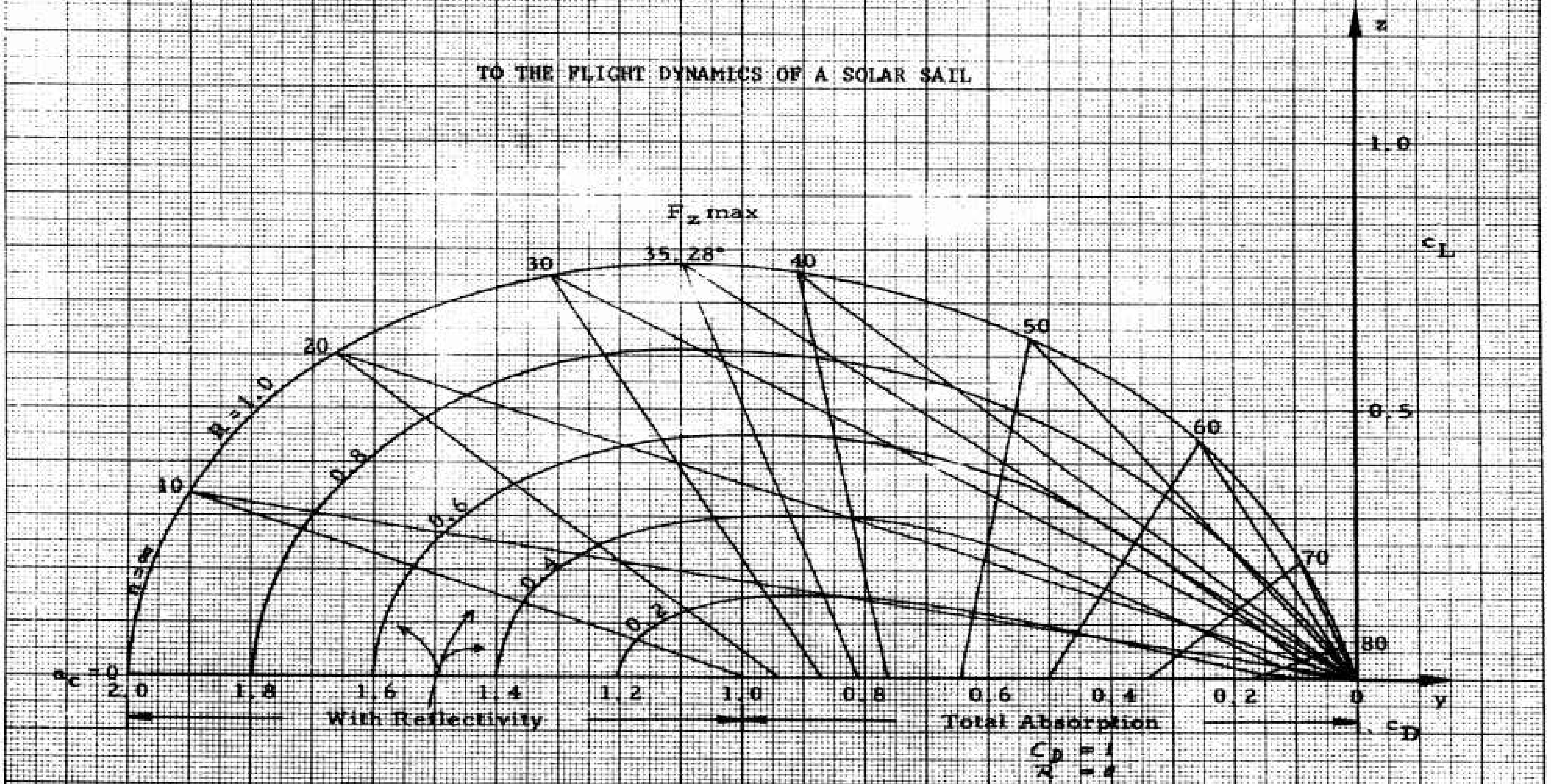
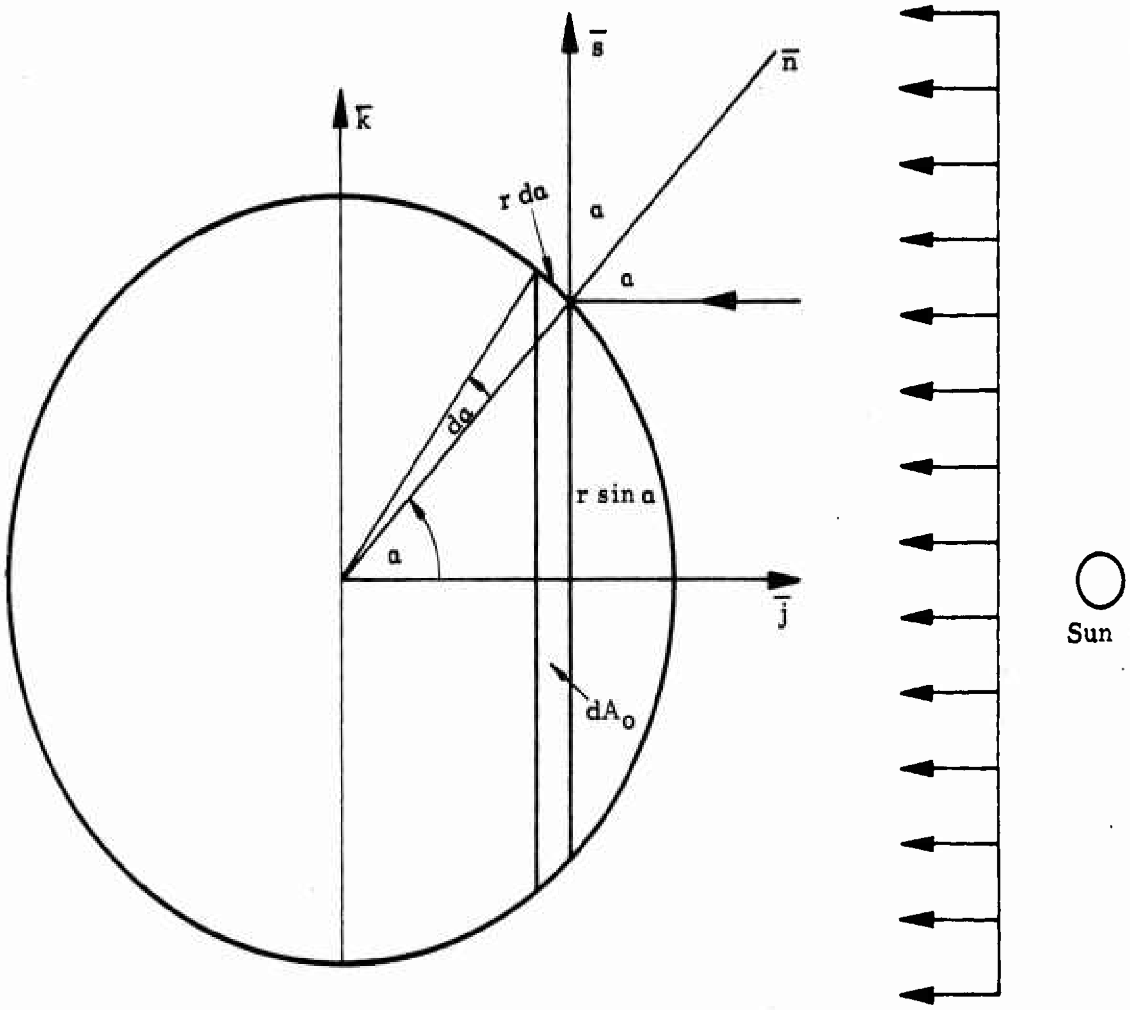
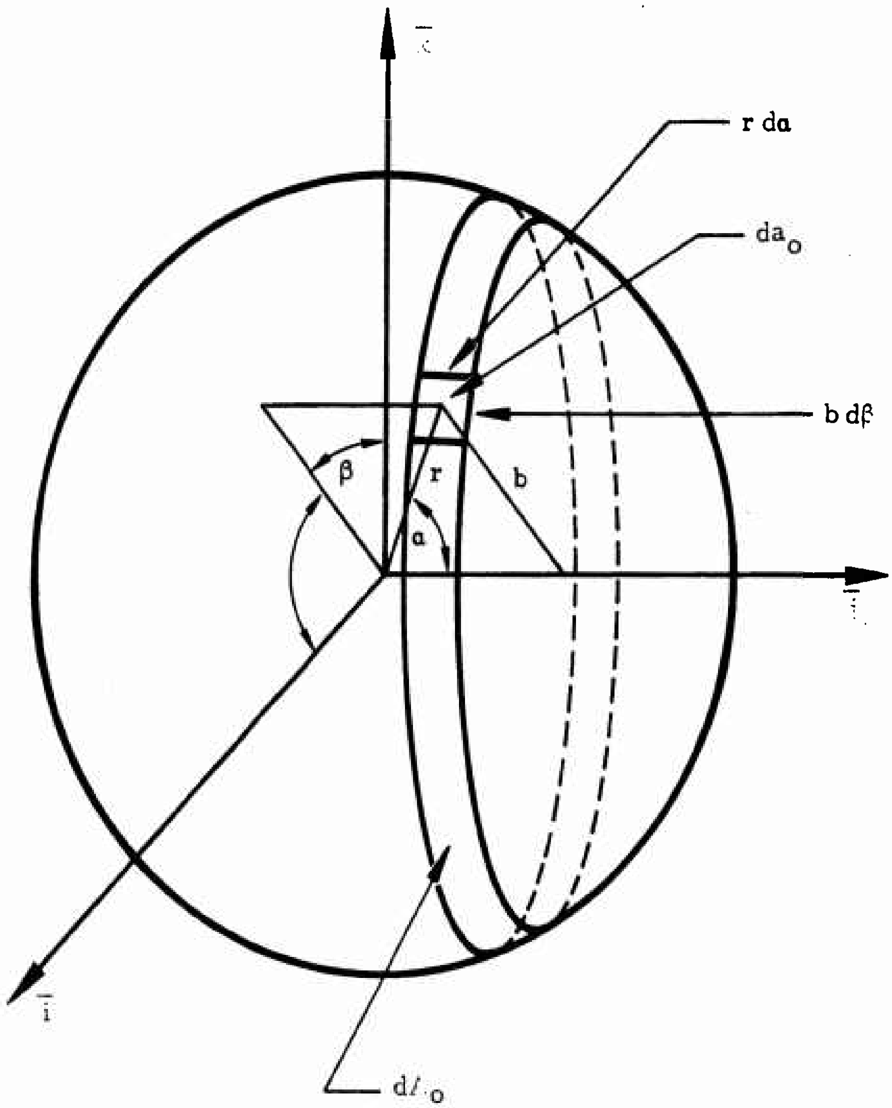



Figure 9

The Variation of the Radiation Force Coefficients c with Angle of Incidence α and Reflectivity R




| Figure 10a | The Radiation Force Acting on a Spherical Body |
|---|---|




| Figure 10b | The Surface Element $da_o$ of a Spherical Body |
|---|---|

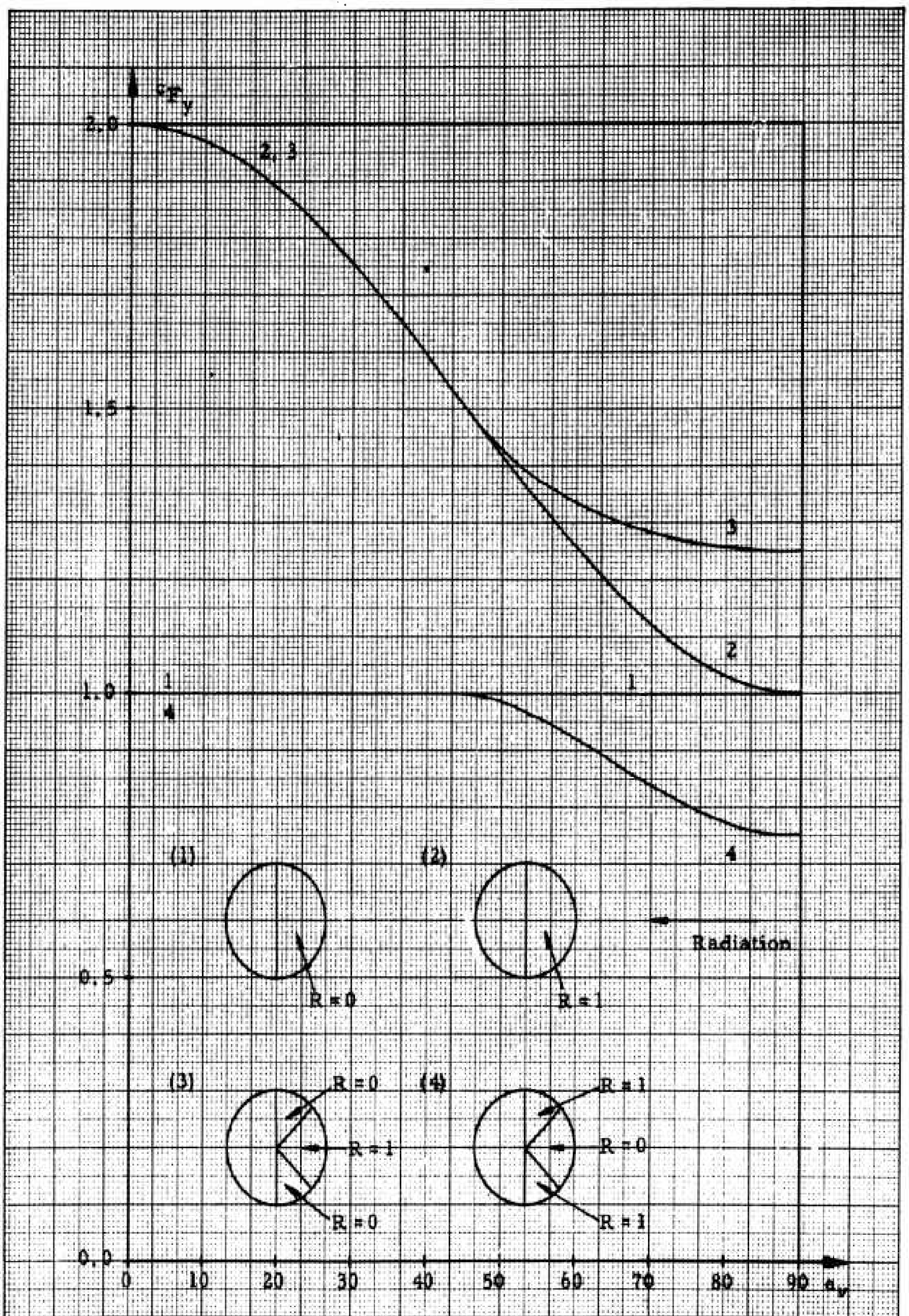



Figure 11 The Radiation Force Coefficient $c_{F_y}$ for a Spherical Surface with Half Opening Angle $a_y$

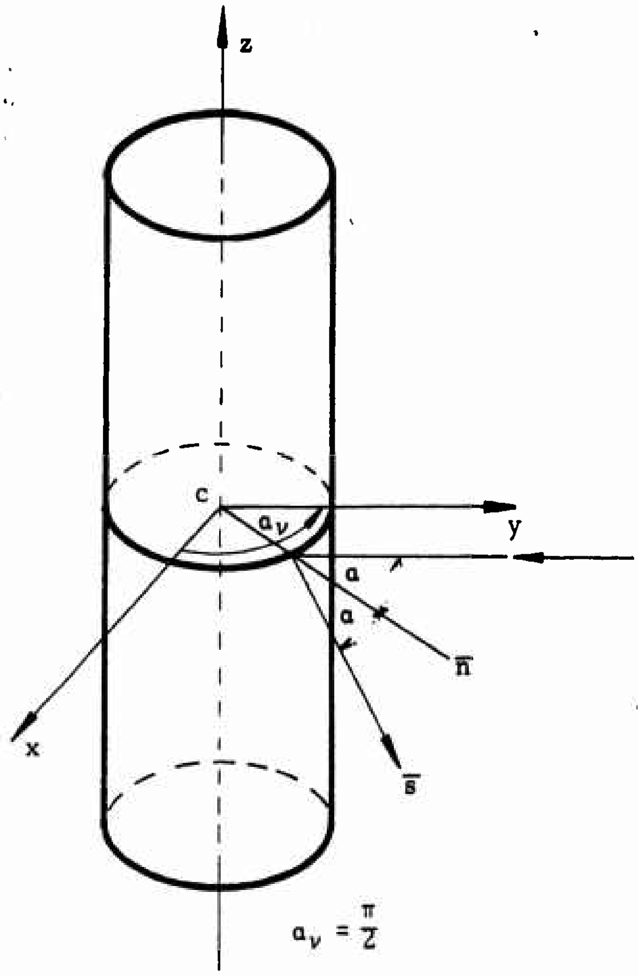



Figure 12a

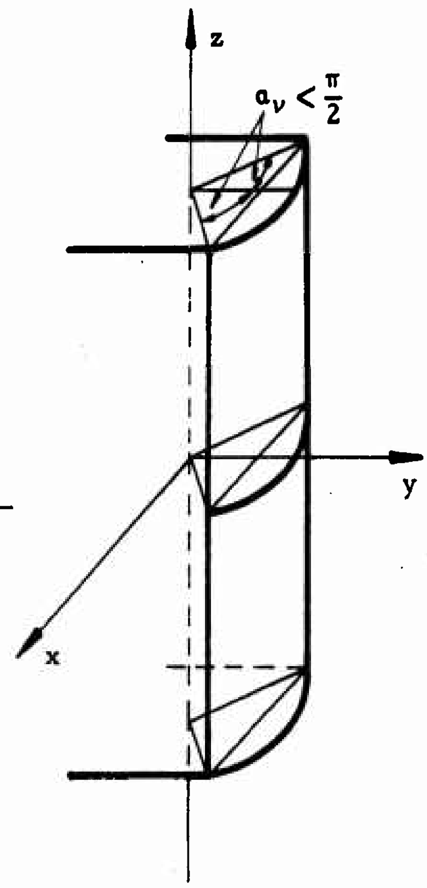



Figure 12b

| | |
|---|---|
| Figure 12a<br>Figure 12b | The Radiation Force Acting on a Cylindrical Body |

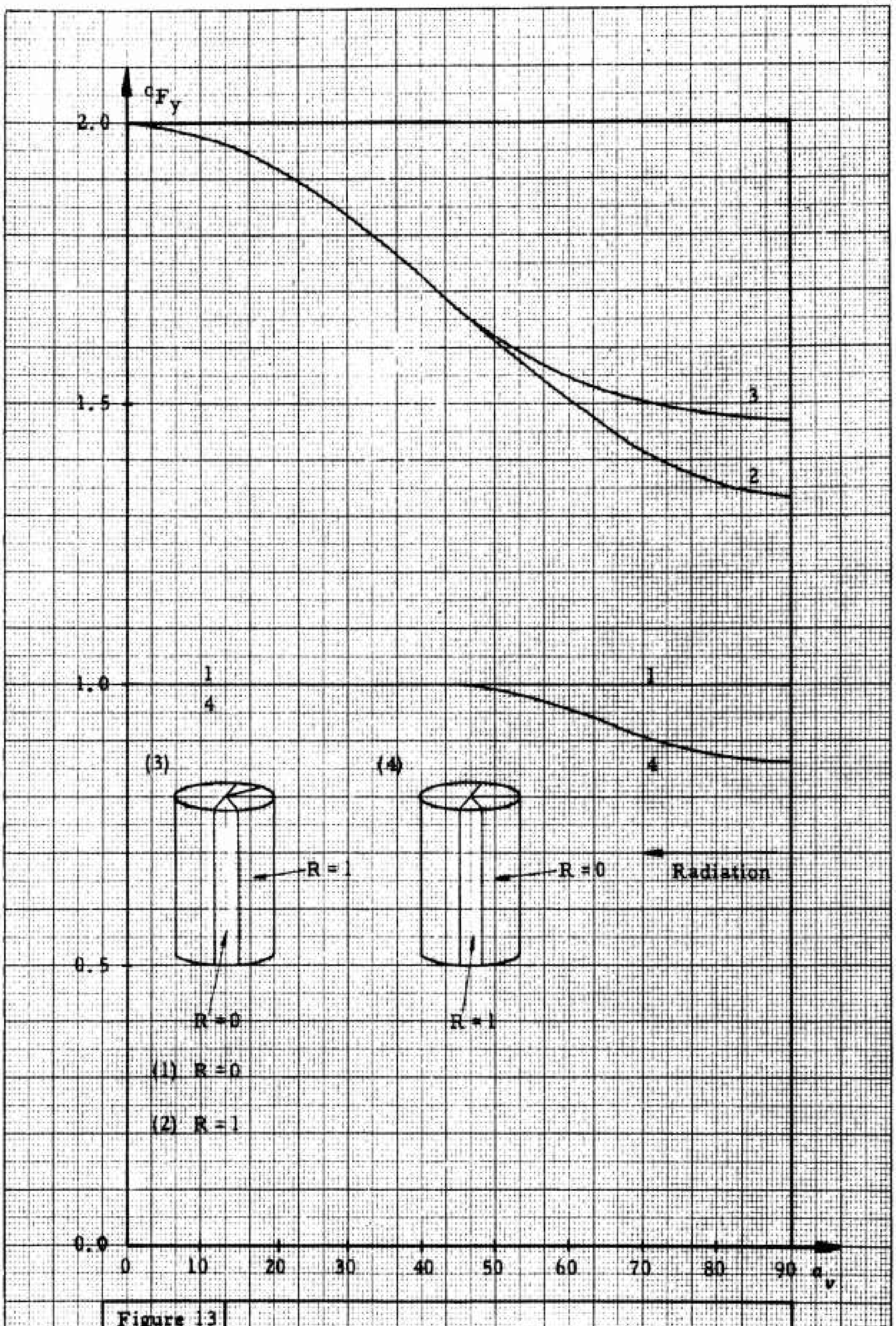
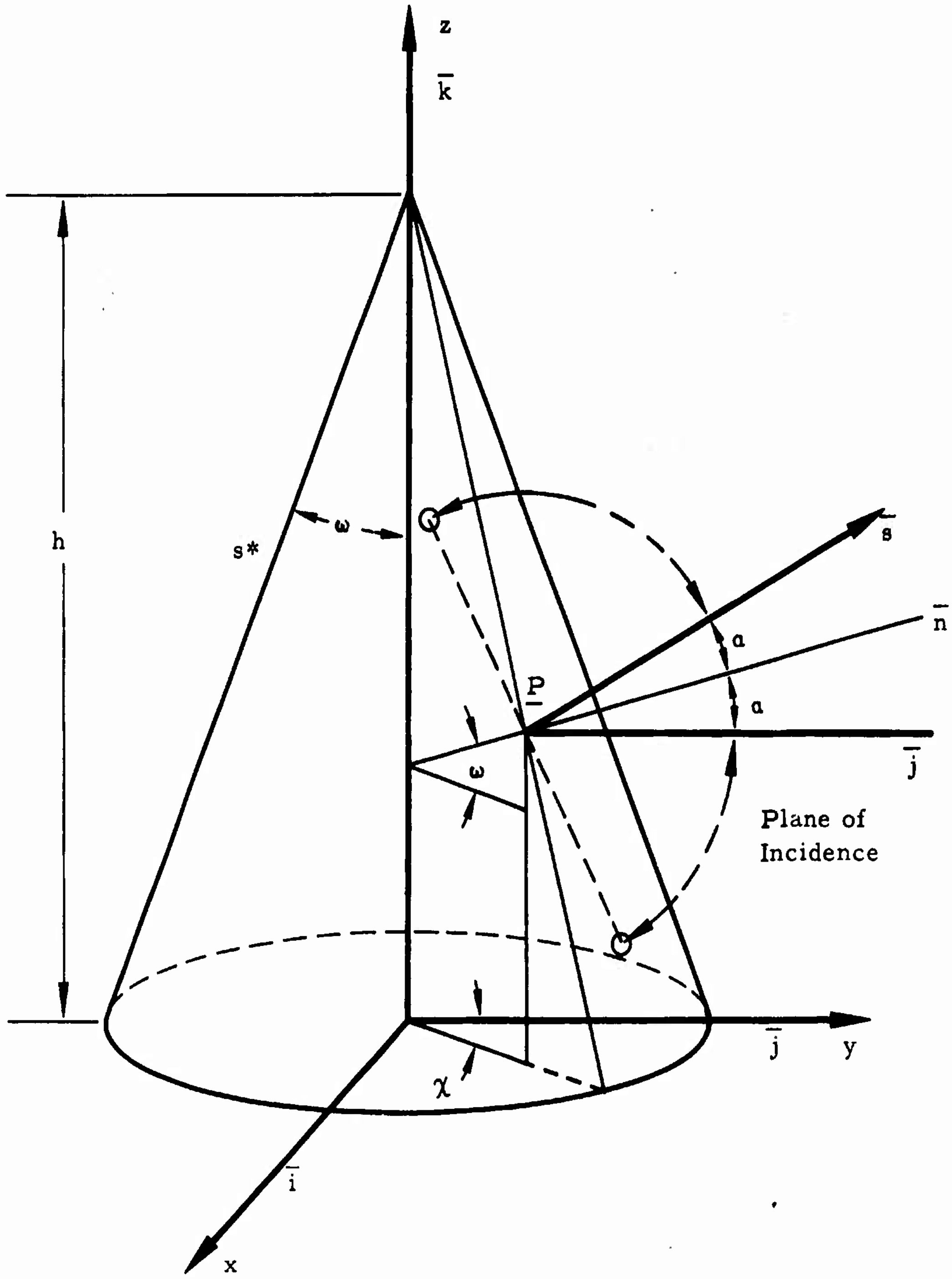



Figure 13 The Radiation Force Coefficient $c_{F_y}$ for a Cylindrical Surface with Half Opening Angle $\alpha_v$




| Figure 14a | The Radiation Force Acting on a Right Circular Cone |
|---|---|

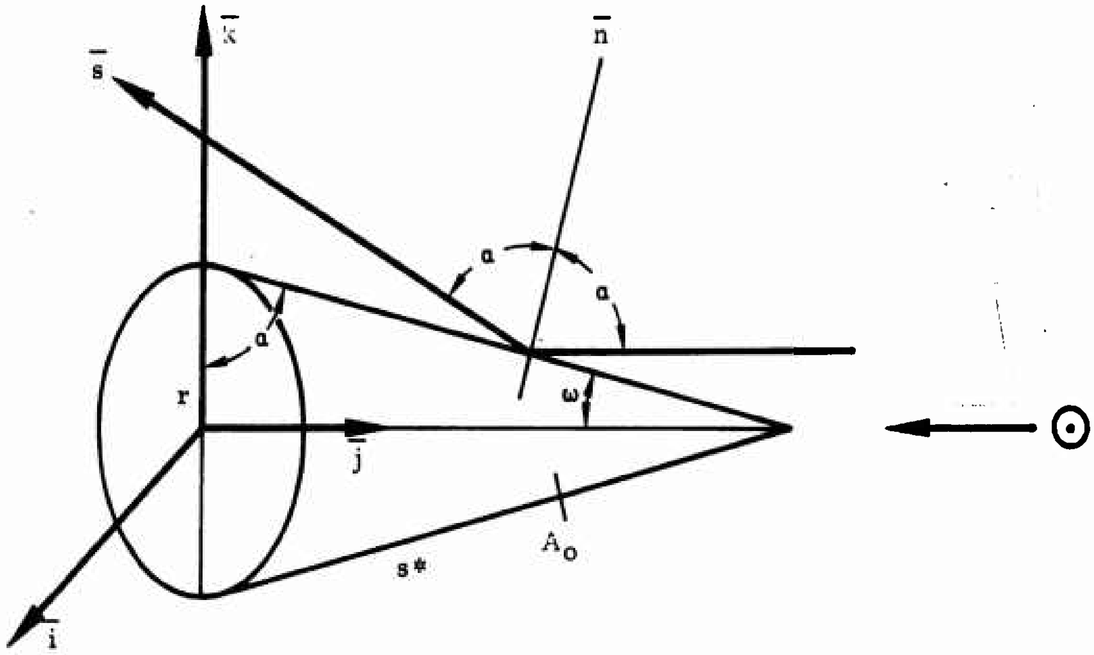



Figure 14b

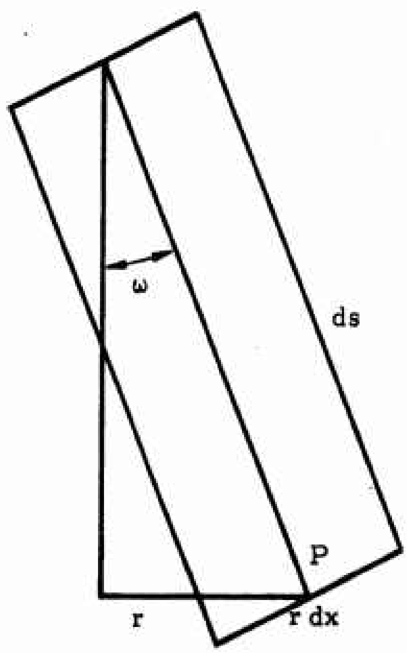



Figure 14c

| | |
|---|---|
| Figure 14b | The Radiation Force Acting on a Right Circular Cone |
| Figure 14c | The Surface Element |

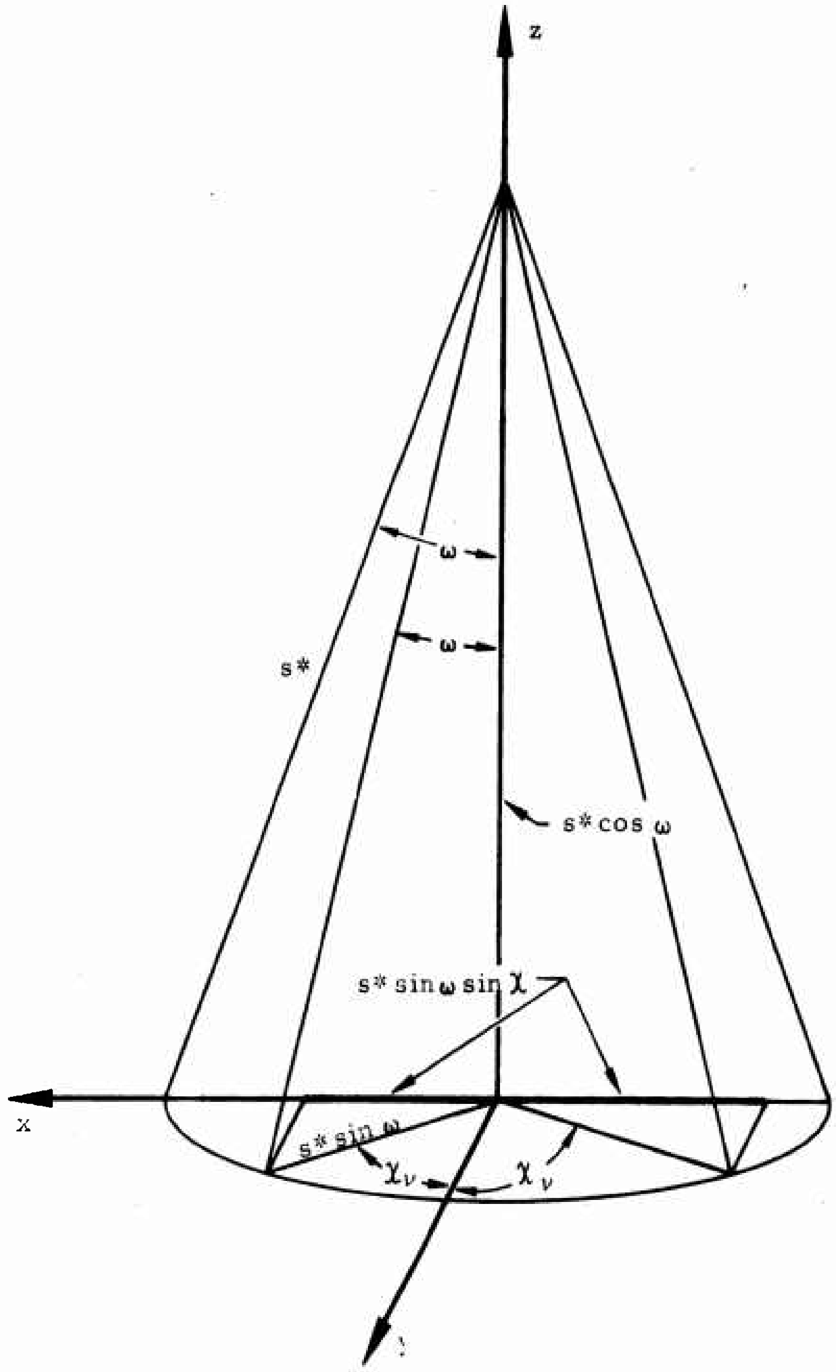
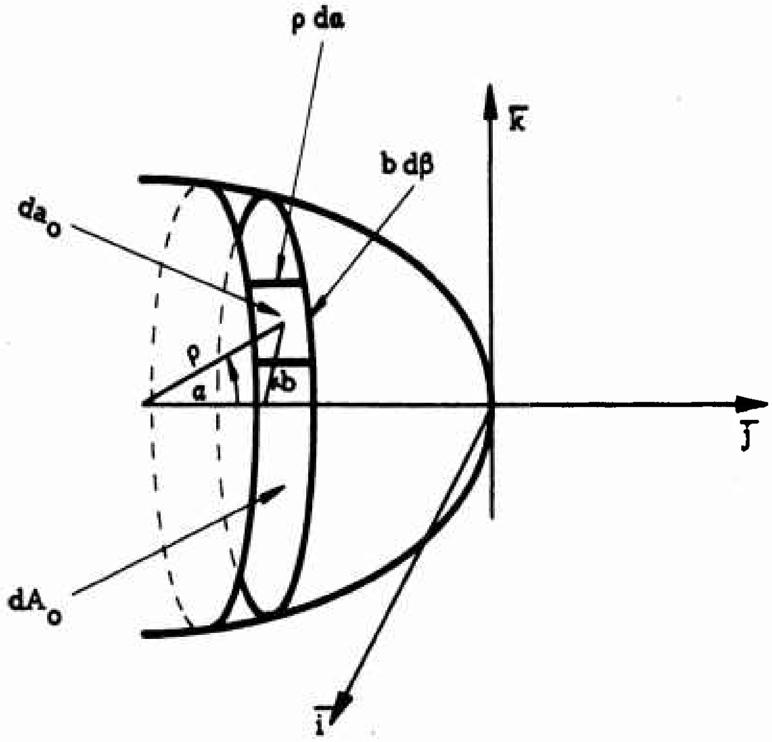
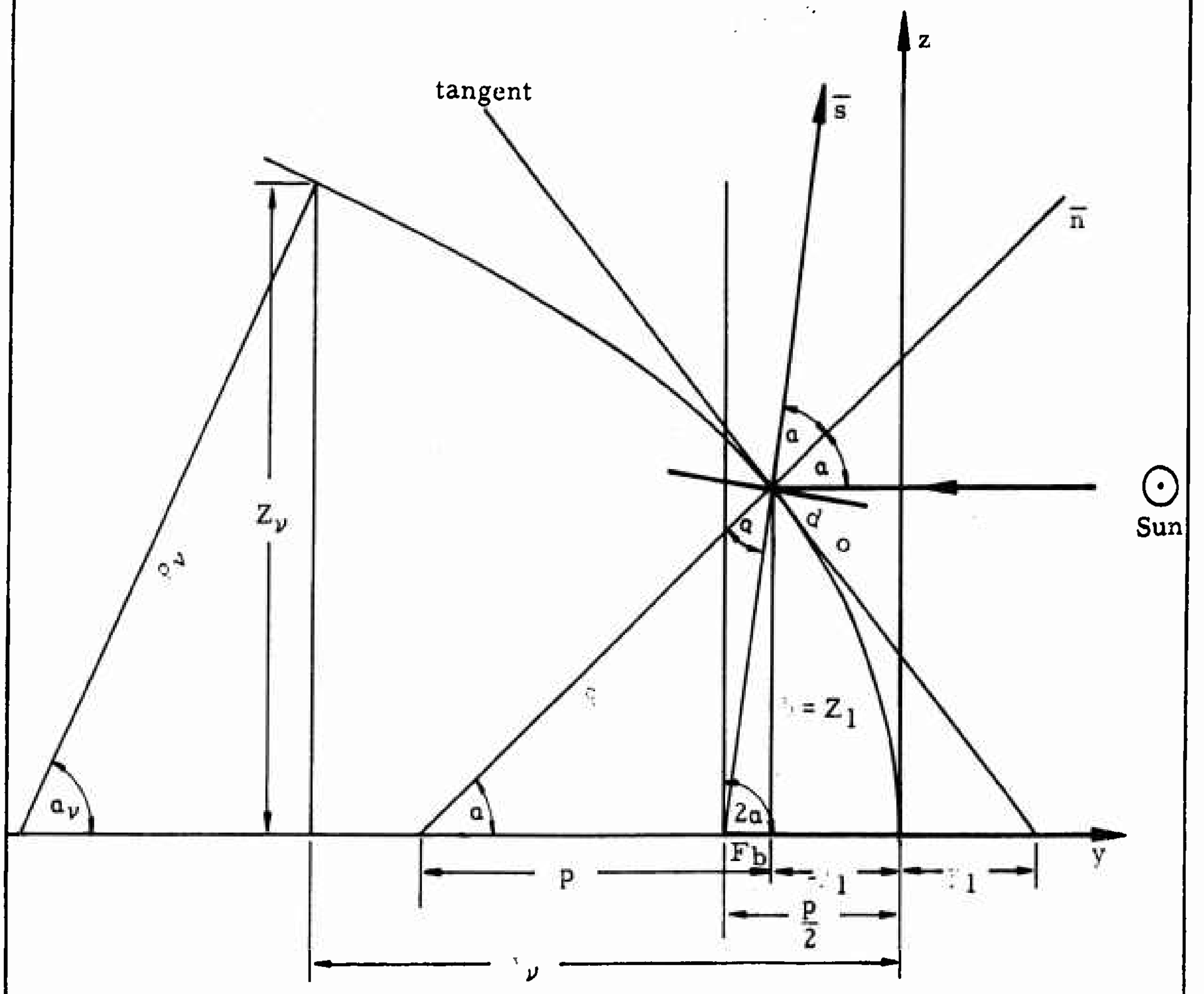



Figure 14d

The Projected Area of a Right Circular Cone




| | |
|---|---|
| Figure 15a | The Surface Element $da_o$ of a Paraboloid |




| Figure 15b | The Radiation Force Acting on a Paraboloid |
|---|---|

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